Math 247A Lecture Notes
Classical Fourier Analysis

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Contents

1 Review: The Fourier Transform

1.1 Properties of the Fourier transform

This class is called "Classical Fourier Analysis," but for the past 20 years, it has been taught more like "Modern Harmonic Analysis." Our treatment will be no different.

Definition 1.1. The **Fourier transform** of a function $f \in L^1(\mathbb{R}^d)$ is given by

$$(\mathcal{F}f)(\xi) = \widehat{f}(\xi) = \int e^{-2\pi i x\cdot\xi} f(x)\,dx.$$

Remark 1.1. By the triangle inequality,

$$\|\widehat{f}\|_{L^\infty} \leq \|f\|_{L^1}.$$

We will prove quantitative results about nice sets of functions and extend these results to more general functions via density arguments. What are our "nice" functions?

Definition 1.2. A C^∞ function $f : \mathbb{R}^d \to \mathbb{C}$ is called a **Schwarz function** if $x^\alpha D^\beta f \in L^\infty$ for all multi-indices $\alpha, \beta \in \mathbb{N}^d$. The vector space of all such functions, $\mathcal{S}(\mathbb{R}^d)$, is called the **Schwarz space**.

This says that all the derivatives of f decay faster than any polynomial. Recall that for a multi-index $\alpha \in \mathbb{N}^d$, we denote

$$|\alpha| = \alpha_1 + \cdots + \alpha_d, \qquad x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_d^{\alpha_d}, \qquad D^\alpha = \frac{\partial^{|\alpha|}}{\partial_{x_1}^{\alpha_1} \cdots \partial_{x_d}^{\alpha_d}}.$$

The Schwarz space is a Fréchet space with the topology generated by the countable family of seminorms $\{\varphi_{\alpha,\beta}\}_{\alpha,\beta\in\mathbb{N}^d}$ with $\varphi_{\alpha,\beta}(f) = \|x^\alpha D^\beta f\|_{L^\infty}$.

Proposition 1.1 (properties of the Fourier transform)**.** *Fix $f \in \mathcal{S}(\mathbb{R}^d)$.*

1. *If $g(x) = f(x-y)$ with $y \in \mathbb{R}^d$ fixed, then $\widehat{g}(\xi) = e^{-2\pi i y\cdot\xi}\widehat{f}(\xi)$.*

   Proof.
   $$\widehat{g}(\xi) = \int e^{-2\pi i x\cdot\xi} f(x-y)\,dx = \int e^{-2\pi i (x+y)\cdot\xi} f(x)\,dx = e^{-2\pi i y\cdot\xi}\widehat{f}(\xi). \qquad \square$$

2. *Let $g(x) = e^{2\pi i x\cdot\eta} f(x)$ for $\eta \in \mathbb{R}^d$ fixed. Then $\widehat{g}(\xi) = \widehat{f}(\xi - \eta)$.*

   Proof.
   $$\widehat{g}(\xi) = \int e^{-2\pi i x(\xi-\eta)} f(x)\,dx = \widehat{f}(\xi-\eta). \qquad \square$$

3. *If $f(x) = f(Tx)$ for $T \in \mathrm{GL}_d(\mathbb{R})$, then $\widehat{f}(\xi) = |\det T|^{-1}\widehat{f}((T^{-1})^\top\xi)$.*

Proof.

$$\begin{aligned}
\widehat{f}(\xi) &= \int e^{-2\pi i x\cdot\xi} f(Tx)\,dx \\
&\overset{y=Tx}{=} |\det T|^{-1}\int e^{-2\pi i T^{-1}y\cdot\xi} f(y)\,dy \\
&= |\det T|^{-1}\int e^{-2\pi i y\cdot(T^{-1})^\top\xi} f(y)\,dy \\
&= |\det T|^{-1}\widehat{f}((T^{-1})^\top\xi).
\end{aligned}$$

□

4. *If $g = \overline{f}$, then $\widehat{g}(\xi) = \overline{\widehat{f}(-\xi)}$.*

5. *If $g = D^\alpha f$ with $\alpha \in \mathbb{N}^d$, then $\widehat{g}(\xi) = (2\pi i\xi)^\alpha\widehat{f}(\xi)$*

Proof. Using integration by parts,

$$\widehat{g}(\xi) = \int e^{-2\pi i x\cdot\xi} D^\alpha f(x)\,dx = (2\pi i\xi)^\alpha\widehat{f}(\xi).$$

□

6. *If $g(x) = x^\alpha f(x)$ for $\alpha \in \mathbb{N}^d$, then*

$$\widehat{g}(\xi) = \frac{1}{(-2\pi i)^{|\alpha|}} D^\alpha\widehat{f}(\xi).$$

Proof.

$$\begin{aligned}
\widehat{g}(\xi) &= \int e^{-2\pi i x\cdot\xi} x^\alpha f(x)\,dx \\
&= \frac{1}{(-2\pi i)^{|\alpha|}}\int e^{-2\pi i x\cdot\xi}(-2\pi i x)^\alpha f(x)\,dx \\
&= \frac{1}{(-2\pi i)^{|\alpha|}} D^\alpha\widehat{f}(\xi).
\end{aligned}$$

□

7. *Let $g = k * f$ for $k \in L^1(\mathbb{R}^d)$. Then $\widehat{g}(\xi) = \widehat{k}(\xi)\widehat{f}(\xi)$.*

Proof.

$$\widehat{g}(\xi) = \int e^{-2\pi i x\cdot\xi}(k * f)(x)\,dx$$

$$
\begin{aligned}
&= \iint e^{-2\pi i x\cdot\xi} k(x-y) f(y)\, dy\, dx \\
&\overset{z=x-y}{=} \iint e^{-2\pi i (z+y)\cdot\xi} k(z) f(y)\, dz\, dy \\
&= \widehat{k}(\xi)\widehat{f}(\xi).
\end{aligned}
$$

□

Remark 1.2. Properties 1, 2, 3, 4, and 7 extend to $f \in L^1(\mathbb{R}^d)$.

Remark 1.3. Property 3 implies that any rotation and/or reflection symmetry $T \in \mathrm{O}_d(\mathbb{R})$ of f is inherited by $\widehat{f}$. Indeed, if $f(x) = f(Tx)$, then

$$\widehat{f}(\xi) = |\det T|^{-1}\widehat{f}((T^{-1})^\top \xi) = \widehat{f}(T\xi).$$

Exercise 1.1. Show that

1. If $f \in \mathcal{S}(\mathbb{R}^d)$, then $\widehat{f} \in \mathcal{S}(\mathbb{R}^d)$.
2. If $f_n \xrightarrow{\mathcal{S}(\mathbb{R}^d)} f$, then $\widehat{f_n} \xrightarrow{\mathcal{S}(\mathbb{R}^d)} \widehat{f}$.

These follow from properties 5 and 6.

1.2 The Riemann-Lebesgue lemma

Lemma 1.1 (Riemann-Lebesgue)**.** *If $f \in L^1(\mathbb{R}^d)$, then $\widehat{f} \in C_0(\mathbb{R}^d)$ (continuous and vanishing at infinity). In particular, $\widehat{f}$ is uniformly continuous.*

Proof. Let $f_n \in \mathcal{S}(\mathbb{R}^d)$ be such that $f_n \xrightarrow{L^1} f$. By the triangle inequality,

$$\|\widehat{f_n} - \widehat{f}\|_{L^\infty} \le \|f_n - f\|_{L^1} \xrightarrow{n\to\infty} 0.$$

Now $\widehat{f_n} \in \mathcal{S}(\mathbb{R}^d) \subseteq C_0(\mathbb{R}^d)$, and $C_0(\mathbb{R}^d)$ is closed in L^∞. So $\widehat{f} \in C_0(\mathbb{R}^d)$. □

1.3 Fourier transform of Gaussians

Lemma 1.2. *Let A be a positive-definite, real-symmetric $d \times d$ matrix. Then*

$$\int e^{-x\cdot Ax} e^{-2\pi i x\cdot\xi}\, dx = (\det A)^{-1/2}\pi^{d/2} e^{-\pi^2 \xi\cdot A^{-1}\xi}.$$

Proof. A real-symmetric, positive-definite matrix is diagonalizable, so there exists an orthogonal $O \in \mathrm{O}_d(\mathbb{R})$ such that $A = O^\top DO$ with $D = \mathrm{diag}(\lambda_1, \ldots, \lambda_\ell)$ with $\lambda_1, \ldots, \lambda_d > 0$. Now

$$x \cdot Ax = x \cdot O^\top DOx = Ox \cdot DOx \overset{y=Ox}{=} y \cdot Dy = \sum \lambda_j y_j^2.$$

We have

$$x \cdot \xi \overset{y=Ox}{=} O^{-1} y \cdot \xi = y \cdot O\xi \overset{\eta=O\xi}{=} y \cdot \eta.$$

So

$$\int e^{-x \cdot Ax} e^{-2\pi i x \cdot \xi}\, dx = \int e^{-\sum(\lambda_j y_j^2 - 2\pi i y_j \eta_j)}\, dy.$$

This is a product of 1-dimensional integrals. Let's look at the 1-dimensional integral

$$\begin{aligned}\int_{\mathbb{R}} e^{-\lambda y^2 - 2\pi i y \eta}\, dy &= \int_{\mathbb{R}} e^{-\lambda(y + \frac{\pi i \eta}{\lambda})^2 - \frac{\pi^2 \eta^2}{\lambda}}\, dy \\ &= \int_{\mathbb{R}} e^{-\lambda y^2} e^{-\pi^2 \eta^2/\lambda}\, dy \\ &= \lambda^{-1/2} \pi^{1/2} e^{-\pi^2 \eta^2/\lambda}.\end{aligned}$$

So we get

$$\begin{aligned}\int e^{-x \cdot Ax} e^{-2\pi i x \cdot \xi}\, dx &= \prod_{j=1}^{d} (\lambda_j^{-1/2} \pi^{1/2} e^{-\pi^2 \eta_j^2/\lambda_j}) \\ &= (\det A)^{-1/2} \pi^{d/2} e^{-\pi^2 \eta \cdot D^{-1} \eta} \\ &= (\det A)^{-1/2} \pi^{d/2} e^{-\pi^2 \xi \cdot O^\top D^{-1} O \xi} \\ &= (\det A)^{-1/2} \pi^{d/2} e^{-\pi^2 \xi \cdot A^{-1} \xi}.\end{aligned}$$

□

Corollary 1.1. $e^{-\pi|x|^2}$ *is n eigenvector of the Fourier transform with eigenvalue 1.*

Proof.

$$[\mathcal{F}(e^{-\pi|x|^2})](\xi) \overset{A=\pi I}{=} e^{-\pi|\xi|^2}.$$

□

1.4 Fourier inversion

Theorem 1.1 (Fourier inversion)**.** *For $f \in \mathcal{S}(\mathbb{R}^d)$, we have*

$$[(\mathcal{F} \circ \mathcal{F}) f](x) = f(-x),$$

or equivalently,

$$f(x) = \int e^{2\pi i x \cdot \xi} \widehat{f}(\xi)\, d\xi.$$

We can think of this as decomposing f into a linear combination of characters with Fourier coefficients.

Proof. We can't use Fubini like we want to because the integrand is not necessarily absolutely integrable. The (standard) trick is to force a Gaussian in there. For $\varepsilon > 0$, let

$$I_\varepsilon(x) = \int e^{-\pi\varepsilon^2|\xi|^2} e^{2\pi i x\cdot\xi} \widehat{f}(\xi)\, d\xi.$$

Then the dominated convergence theorem tells us that $I_\varepsilon(x) \to \int e^{2\pi i x\cdot\xi} \widehat{f}(\xi)\, d\xi$ as $\varepsilon \to 0$. $\square$

Next time, we will complete the proof.

2 Fourier Inversion and Plancherel's Theorem

2.1 Fourier inversion

Theorem 2.1 (Fourier inversion)**.** *For $f \in \mathcal{S}(\mathbb{R}^d)$, we have*

$$[(\mathcal{F} \circ \mathcal{F})f](-x) = f(x),$$

or equivalently,

$$f(x) = \int e^{2\pi i x \cdot \xi} \widehat{f}(\xi)\, d\xi.$$

We can think of this as decomposing f into a linear combination of characters with Fourier coefficients.

Proof. We can't use Fubini like we want to because the integrand is not necessarily absolutely integrable. The (standard) trick is to force a Gaussian in there. For $\varepsilon > 0$, let

$$I_\varepsilon(x) = \int e^{-\pi\varepsilon^2 |\xi|^2} e^{2\pi i x \cdot \xi} \widehat{f}(\xi)\, d\xi.$$

Then the dominated convergence theorem tells us that $I_\varepsilon(x) \to \int e^{2\pi i x\cdot\xi} \widehat{f}(\xi)\, d\xi$ as $\varepsilon \to 0$. On the other hand,

$$\begin{aligned} I_\varepsilon(x) &= \iint e^{-\pi\varepsilon^2|\xi|^2} e^{2\pi i x\cdot\xi} e^{-2\pi i y\cdot\xi} f(y)\, dy\, d\xi \\ &= \int f(y) \int e^{-\pi\varepsilon^2|\xi|^2} e^{-2\pi i(y-x)\cdot\xi}\, d\xi\, dy \end{aligned}$$

Use our lemma from last time with the linear transformation $A = \pi\varepsilon^2 I$:

$$\begin{aligned} &= \int f(y)(\pi\varepsilon^2)^{-d/2} \pi^{d/2} e^{-\pi^2 (y-x)\frac{1}{\pi\varepsilon^2}(y-x)}\, dy \\ &= \int \varepsilon^{-d} e^{-\frac{\pi}{\varepsilon^2}|x-y|^2} f(y)\, dy. \end{aligned}$$

Note that $\int \varepsilon^{-d} e^{-\frac{\pi}{\varepsilon^2}|x|^2}\, dx = \int e^{-\pi|x|^2}\, dx$.

$$\xrightarrow{\varepsilon\to 0} f(x).$$

To show this convergence, we have $\int \varepsilon^{-d} e^{-\frac{\pi}{\varepsilon^2}|x-y|^2} f(y)\, dy - f(x) = \int \varepsilon^{-d} e^{-\frac{\pi}{\varepsilon^2}|x-y|^2}\, dx [f(y) - f(x)]\, dy$. For $\eta > 0$, there is a $\delta(\eta) > 0$ such that $|f(y) - f(x)| < \eta$ whenever $|x - y| < \delta$. Then

$$\left| \int_{|x-y|<\delta} \varepsilon^{-d} e^{\frac{\pi}{\varepsilon^2}|x-y|^2} [f(y) - f(x)]\, dy \right| \leq \eta \int_{|x-y|<\delta} \varepsilon^{-d} e^{\frac{\pi}{\varepsilon^2}|x-y|^2}\, dy \leq \eta,$$

$$\left| \int_{|x-y|>\delta} \varepsilon^{-d} e^{\frac{\pi}{\varepsilon^2}|x-y|^2} [f(y) - f(x)]\, dy \right| \le 2\|f\|_{L^\infty} \int_{|y|>\delta} \varepsilon^{-d} e^{-\frac{\pi}{\varepsilon^2}|y|^2}\, dy$$
$$\le 2\|f\|_{L^\infty} \int_{|y|>\delta} e^{-\pi|y|^2}\, dy$$
$$\lesssim \|f\|_{L^\infty} e^{-\pi \frac{\delta^2}{2\varepsilon^2}}$$
$$\xrightarrow{\varepsilon\to 0} 0.$$

First pick $\eta \ll 1$. Then choose $\varepsilon = \varepsilon(\delta) = \varepsilon(\eta) \ll 1$. □

Corollary 2.1. *The Fourier transform is a homeomorphism on $\mathcal{S}(\mathbb{R}^d)$.*

2.2 Plancherel's theorem

Lemma 2.1. *For $f, g \in \mathcal{S}(\mathbb{R}^d)$, we have*

$$\int \widehat{f}(\xi)\overline{\widehat{g}(\xi)}\, d\xi = \int f(x)\overline{g(x)}\, dx.$$

In particular,

$$\|\widehat{f}\|_{L^2} = \|f\|_{L^2}.$$

so $\mathcal{F}$ is an isometry in L^2 on $\mathcal{S}(\mathbb{R}^d)$.

Proof. For $h \in \mathcal{S}(\mathbb{R}^d)$,

$$\int \widehat{f}(\xi)h(\xi)\, d\zeta = \iint e^{-2\pi i x\cdot\xi} f(x)h(\xi)\, dx\, d\xi$$
$$= \int f(x)\widehat{h}(x)\, dx.$$

Now let $h = \overline{\hat{g}}$. Then $(\mathcal{F}h)(x) = \overline{\mathcal{F}(\widehat{g})(-x)} = \overline{g(x)}$. □

Theorem 2.2 (Plancherel)**.** *The Fourier transform extends from $\mathcal{S}(\mathbb{R}^d)$ to a unitary map on $L^2(\mathbb{R}^d)$.*

Proof. Fix $f \in L^2(\mathbb{R}^d)$. To define the Fourier transform on $\mathcal{F}$, let $f_n \in \mathcal{S}(\mathbb{R}^d)$ be such that $f_n \xrightarrow{L^2} f$. Since $\mathcal{F}$ is an isometry in L^2 on $\mathcal{S}(\mathbb{R}^d)$, $\|\widehat{f}_n - \widehat{f}_m\|_{L^2} = \|f_n - f_m\|_{L^2} \xrightarrow{n,m\to\infty} 0$. So $\{\widehat{f}_n\}_{n\ge 1}$ is Cauchy and hence convergent in $L^2(\mathbb{R}^d)$. Let $\widehat{f}$ be the L^2 limit of the $\widehat{f}_n$.

We claim that $\widehat{f}$ does not depend on the sequence $\{f_n\}_{n\ge1}$. Let $\{g_n\}_{n\ge1} \subseteq \mathcal{S}(\mathbb{R}^d)$ be another sequence such that $g_n \xrightarrow{L^2} f$. Let

$$h_n = \begin{cases} f_k & n = 2k-1 \\ g_k & n = 2k. \end{cases}$$

We have that $\{h_n\} \subseteq \mathcal{S}(\mathbb{R}^d)$, and $h_n \xrightarrow{L^2} f$. By the same argument as before, $\{\widehat{h}_n\}_{n\geq 1}$ converges in L^2. This means that $\lim_n \widehat{h}_n = \lim_n \widehat{f}_n = \lim_n \widehat{g}_n$.

We now claim that $\|\widehat{f}\|_2 = \|f\|_2$ for all $f \in L^2(\mathbb{R}^d)$; i.e. $\mathcal{F}$ is an isometry on L^2. Indeed,

$$\|\widehat{f}\|_2 = \lim_n \|\widehat{f}_n\|_2 = \lim \|f_n\|_2 = \|f\|_2.$$

Remark 2.1. This is not yet enough to show that $\mathcal{F}$ is unitary. In infinite dimensions, isometries need not be unitary. For example, take $T : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ be $T(a_1, a_2, \dots) = (0, a_1, a_2, \dots)$. Then

$$\langle T(a_1, a_2, \dots), (b_1, b_2, \dots)\rangle = \sum_{n\geq 1} a_n b_{n+1} = \langle (a_1, a_2, \dots), (b_2, b_3, \dots)\rangle,$$

so $T^*(a_1, a_2, \dots) = (a_2, a_3, \dots)$. So $T^*T = \text{id}$, but $TT^* \neq \text{id}$. What we need to get an isometry is surjectivity.

We claim that $\mathcal{F} : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ is onto. We will show that $\text{Ran}(\mathcal{F})$ is closed in $L^2(\mathbb{R}^d)$. As $\text{Ran}(\mathcal{F}) \supseteq \mathcal{S}(\mathbb{R}^d)$, this will give $L^2(\mathbb{R}^d) = \overline{\mathcal{S}(\mathbb{R}^d)}^{L^2} \subseteq \overline{\text{Ran}(\mathcal{F})}^{L^2} = \text{Ran}(\mathcal{F})$. Let $g \in \overline{\text{Ran}(\mathcal{F})}^{L^2}$. Then there exist $f_n \in L^2$ such that $\widehat{f}_n \xrightarrow{L^2} g$. $\mathcal{F}$ is an isometry on $L^2(\mathbb{R}^d)$, so $\|f_n - f_m\|_2 = \|\widehat{f}_n - \widehat{f}_m\|_2 \xrightarrow{n,m\to\infty} 0$. So $\{f_n\}_{n\geq 1}$ converges in L^2 to some f. Then $g = \widehat{f}$ because

$$\|\widehat{f} - \widehat{f}_n\|_2 = \|f - f_n\|_2 \xrightarrow{n\to\infty} 0.$$

By the uniqueness of limits, we get $g = \widehat{f}$. So we get $g = \widehat{f} \in \text{Ran}(\mathcal{F})$. ☐

2.3 The Hausdorff-Young inequality

Theorem 2.3 (Hausdorff-Young)**.** *For $f \in \mathcal{S}(\mathbb{R}^d)$,*

$$\|\widehat{f}\|_{p'} \leq \|f\|_p, \qquad \forall 1 \leq p \leq 2,$$

where $1/p + 1/p' = 1$.

Proof. This follows from interpolation, as we have $\mathcal{F} : L^1 \to L^\infty$ with $\|\widehat{f}\|_{L^\infty} \leq \|f\|_{L^1}$ and $\mathcal{F} : L^2 \to L^2$ with $\|\widehat{f}\|_{L^2} = \|f\|_{L^2}$. ☐

Remark 2.2. As in the proof of Plancherel's theorem, we can use Hausdorff-Young to extend the Fourier transform from $\mathcal{S}(\mathbb{R}^d)$ to $L^p(\mathbb{R}^d)$ for any $1 \leq p \leq 2$.

Note that the Riemann-Lebesgue lemma gives that for $f \in L^1(\mathbb{R}^d)$, $\widehat{f} \in C_0(\mathbb{R}^d)$. So we can think of evaluating the Fourier transform at a single point or on a measure 0 set, such as a plane in $\mathbb{R}^3$. The **restriction problem** asks: For which values of p can we make sense of the Fourier transform on measure 0 sets, such as a parabaloid or a cone? This is important in PDE, and it is very hard (still open!).

The next theorem says that the Hausdorff-Young inequality is the best we can do.

Theorem 2.4. *If $\|\widehat{f}\|_{L^q} \leq \|f\|_{L^p}$ for some $1 \leq p, q \leq \infty$ and all $f \in \mathcal{S}(\mathbb{R}^d)$, then necessarily, $q = p'$ and $1 \leq p \leq 2$.*

Proof. For $f \in \mathcal{S}(\mathbb{R}^d)$ with $f \not\equiv 0$, define $f_\lambda(x) = f(x/\lambda)$ for $\lambda > 0$. Then $\|f_\lambda\|_p = \lambda^{d/p}\|f\|_p$. We also have

$$\widehat{f_\lambda}(\xi) = \int e^{-2\pi i x\cdot\xi} f(x/\lambda)\, dx = \lambda^d \widehat{f}(\lambda\xi),$$

so $\|\widehat{f_\lambda}\|_q = \lambda^{d-d/q}\|\widehat{f}\|_q$. Then $\|\widehat{f_\lambda}\|_q \leq \|f_\lambda\|_p$ if and only if $\lambda^{d-d/q}\|\widehat{f}\|_q \leq \lambda^{d/p}\|f\|_p$, so $\lambda^{d(1-1/q-1/p)}\|\widehat{f}\|_q \leq \|f\|_p$. Letting $\lambda \to 0$ or $\lambda \to \infty$, we conclude that $1 - 1/q - 1/p = 1$. So we get $q = p$. $\square$

Next time, we will prove the remaining portion of this theorem, that $1 \leq p \leq 2$.

3 The Littlewood Principle and Lorentz Spaces

3.1 The Littlewood principle and optimality of the Hausdorff-Young inequality

Last time we were proving the following theorem.

Theorem 3.1. *If $\|\widehat{f}\|_{L^q} \leq \|f\|_{L^p}$ for some $1 \leq p, q \leq \infty$ and all $f \in \mathcal{S}(\mathbb{R}^d)$, then necessarily, $q = p'$ and $1 \leq p \leq 2$.*

We have already proven the first statement. To prove the second we will use the **Littlewood principle**: "A translation invariant operator does not improve decay." So if $T : L^p \to L^q$, then $q \geq p$. This is not a theorem but a general principle.

Say we have a bump function at 0 and we translate it far away. Keep doing this (N times), and let f be the superposition of all the bump functions. If we apply T to f, since T is translation invariant, we will get N translated copies of the modified bump. Then $\|f\|_{L^p} \sim N^{1/p}$, while $\|Tf\|_{L^q} \sim N^{1/q}$. Then we need $N^{1/q} \lesssim N^{1/p}$. Letting $N \to \infty$, we get $1/q \leq 1/p$, so $p \leq q$.

The Fourier transform is not translation invariant, however. And the Fourier transform of a compactly supported function no longer has compact support. However, we can use the fast decay of the Gaussian to achieve the same effect.

Proof. Let $\varphi(x) = e^{-\pi|x|^2}$. For $1 \leq k \leq N$ and $\alpha \gg 1$, define

$$\varphi_k(x) = e^{2\pi i x \cdot \alpha k e_1} \varphi(x - \alpha k e_1).$$

Then

$$\widehat{\varphi}_k(\xi) = e^{-2\pi i \alpha k \xi_1} \widehat{\varphi}(\xi - \alpha k e_1).$$

Let $f = \sum_{k=1}^N \varphi_k$ and $S = \bigcup_{j=1}^N \{x : |x - \alpha_j e_1| \leq \alpha/10\}$. Then

$$\|f\|_{L^p} = \|f\|_{L^p(S)} + \|f\|_{L^p(\mathbb{R}^d \setminus S)}.$$

We can bound each of these by

$$\|f\|_{L^p(\mathbb{R}^d \setminus S)} \leq \sum_{k=1}^N \|\varphi_k\|_{L^p(\mathbb{R}^d \setminus S)} \lesssim N\alpha^{-100},$$

$$\|f\|_{L^p(S)}^p \sim \sum_{j=1}^N \underbrace{\left\| \sum_{k=1}^N \varphi_k \right\|_{L^p(|x - \alpha j e_1| \leq \alpha/10)}}_{\sim 1 + O(\sum_{k \neq j} (\alpha|k-j|)^{-100})} \sim N(1 + O(\alpha^{-100}))$$

because $|x - \alpha k e_1| \geq |\alpha(j-k)e_1| - |x - \alpha j e_1| \geq \alpha|j-k| - \alpha/10 \geq (\alpha/2)|j-k|$.

Taking $\alpha \gg 1$, we get $\|f\|_{L^p} \sim N$. Similarly,

$$\|\widehat{f}\|_{L^{p'}} \sim N^{1/p'}$$

We need $N^{1/p'} \le N^{1/p}$ for all $N \ge 1$. This means that $1/p' \le 1/p$, so $p \le p'$. So $1 \le p \le 2$. $\square$

3.2 Weak L^p and Lorentz spaces

Definition 3.1. For $1 \le p < \infty$ and $f : \mathbb{R}^d \to \mathbb{C}$, define

$$\|f\|^*_{L^p_{\text{weak}}} = \sup_{\lambda > 0} \lambda |\{x : |g(x)| > \lambda\}|^{1/p}.$$

The **weak L^p space** is the set of measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ for which $\|f\|^*_{L^p_{\text{weak}}} < \infty$. We denote it by $L^p_{\text{weak}}(\mathbb{R}^d)$.

Example 3.1. $f(x) = |x|^{d/p}$ is in $L^p_{\text{weak}} \setminus L^p$. We have

$$\|f\|^*_{L^{\text{weak}}} = \sup_{\lambda > 0} \lambda |\{x : |x|^{-d/p} > \lambda\}|^{1/p} \sim \sup_{\lambda > 0} \lambda (\lambda^{-p})^{1/p} \sim 1.$$

Remark 3.1. We will show that the weak L^p "norm" is a quasinorm (not a norm) and that is why we append $*$ to the usual norm notation.

By comparison, for $1 \le p < \infty$,

$$\begin{aligned}
\|f\|^p_{L^p} &= \int |f(x)|^p \, dx \\
&= \int \int_0^{|f(x)|} p\lambda^{p-1} \, d\lambda \, dx \\
&= \int_0^\infty p\lambda^{p-1} |\{x : |f(x)| > \lambda\}| \, d\lambda \\
&= p \int_0^\infty \lambda^p |\{x : |f(x)| > \lambda\}| \frac{1}{\lambda} \, d\lambda.
\end{aligned}$$

So we can write

$$\|f\|_{L^p} = p^{1/p} \|\lambda |\{x : |f(x)| > \lambda\}|^{1/p}\|_{L^p((0,\infty), \frac{d\lambda}{\lambda})}.$$

With the convention that $p^{1/\infty} = 1$, we also have

$$\|f\|^*_{L^p_{\text{weak}}} = p^{1/\infty} \|\lambda |\{x : |f(x)| > \lambda\}|^{1/p}\|_{L^\infty((0,\infty), \frac{d\lambda}{\lambda})}.$$

Can we do this to L^p spaces for other exponents?

Definition 3.2. For $1 \le p < \infty$ and $1 \le q \le \infty$, the **Lorentz space** $L^{p,q}(\mathbb{R}^d)$ is the set of measurable functions $f : \mathbb{R}^d \to \mathbb{C}$ for which

$$\|f\|^*_{L^{p,q}(\mathbb{R}^d)} = p^{1/q} \|\lambda |\{x : |f(x)| > \lambda\}|^{1/p}\|_{L^q((0,\infty), \frac{d\lambda}{\lambda})} < \infty.$$

Note that $L^{p,p} = L^p$ and $L^{p,\infty} = L^p_{\text{weak}}$.

Lemma 3.1. $\|f\|^*_{L^{p,q}(\mathbb{R}^d)}$ *is a quasinorm.*

Proof. If $\|f\|^*_{L^{p,q}} = 0$, then $f = 0$ a.e. For $a \neq 0$,

$$\begin{aligned} \|af\|^*_{L^{p,q}} &= p^{1/q} \|\lambda |\{x : |af(x)| > \lambda\}|^{1/p}\|_{L^q(\frac{d\lambda}{\lambda})} \\ &= p^{1/q}|a| \left\| \frac{\lambda}{|a|} |\{x : |f(x)| > \lambda/|a|\}|^{1/p} \right\|_{L^q(\frac{d\lambda}{\lambda})} \\ &= |a| \|f\|^*_{L^{p,q}}. \end{aligned}$$

For the "triangle inequality," we have

$$\begin{aligned} \|f+g\|^*_{L^{p,q}} &= p^{1/q} \|\lambda |\{x : |f(x) + g(x)| > \lambda\}|^{1/p}\|_{L^q(\frac{d\lambda}{\lambda})} \\ &\le p^{1/q} \|\lambda [|\{x : |f(x)| > \lambda/2\}| + |\{x : |f(x)| > \lambda/2\}|]^{1/p}\|_{L^q(\frac{d\lambda}{\lambda})} \end{aligned}$$

By the concavity of fractional powers, we get

$$\begin{aligned} &\le p^{1/q} \left[\left\| \frac{\lambda}{2} |\{x : |f(x)| > \lambda/2\}|^{1/p} \right\|_{L^q(\frac{d\lambda}{\lambda})} + \left\| \frac{\lambda}{2} |\{x : |f(x)| > \lambda/2\}|^{1/p} \right\|_{L^q(\frac{d\lambda}{\lambda})} \right] \\ &\le 2[\|f\|^*_{L^{p,q}} + \|g\|^*_{L^{p,q}}]. \end{aligned}$$

□

Remark 3.2. We will show that for $1 < p < \infty$ and $1 \le q \le \infty$, there exists a norm equivalent to this quasinorm. For $p = 1$ and $q \neq 1$, no such norm exists. Nonetheless, in this latter case, there is a metric that generates the same topology. In all cases, $L^{p,q}(\mathbb{R}^d)$ is complete.

Proposition 3.1. *For* $f \in L^{p,q}(\mathbb{R}^d)$, *decompose* $f = \sum_{m \in \mathbb{Z}} f_m$ *by defining* $f_m(x) = f(x)\mathbb{1}_{\{2^m \le |f(x)| < 2^{m+1}\}}(x)$. *Then*

$$\|f\|^*_{L^{p,q}} \sim \left\| \|f_m\|_{L^p(\mathbb{R}^d)} \right\|_{\ell^q_m(\mathbb{Z})}.$$

In particular, $L^{p,q_1} \subseteq L^{p,q_2}$ *whenever* $q_1 \le q_2$.

4 Relationships Between The Lorentz Quasinorms and L^p Norms

4.1 Order of growth of Lorentz quasinorms in terms of L^p and ℓ^q

Last time, we had the quasinorm

$$\|f\|^*_{L^{p,q}(\mathbb{R}^d)} = p^{1/q} \|\lambda |\{x : |f(x)| > \lambda\}|^{1/p}\|_{L^q((0,\infty),\frac{d\lambda}{\lambda})}$$

Remark 4.1. If $|g| \le |f|$, then $\|g\|^*_{L^{p,q}} \le \|f\|^*_{L^{p,q}}$.

Proposition 4.1. *If $f \in L^{p,q}(\mathbb{R}^d)$ for $1 \le p < \infty$ and $1 \le q \le \infty$, write $f = \sum_{m\in\mathbb{Z}} f_m$, where $f_m(x) = f(x)\mathbb{1}_{\{x:2^m \le |f(x)| < 2^{m+1}\}}(x)$. Then*

$$\|f\|^*_{L^{p,q}} \sim \Big\| \|f_m\|_{L^p(\mathbb{R}^d)} \Big\|_{\ell^q_m(\mathbb{Z})}.$$

Proof. Both sides only concern $|f|$, so it suffices to prove this for $f \ge 0$. Then

$$2^m \mathbb{1}_{\{2^m \le f(x) < 2^{m+1}\}} \le f_m < 2^{m+1} \mathbb{1}_{\{2^m \le f(x) < 2^{m+1}\}}.$$

Thus, by our previous remark, we may assume that $f = \sum_{m\in\mathbb{Z}} 2^m \mathbb{1}_{F_m}$, where F_m are measurable, pairwise disjoint sets.

$$\begin{aligned}
(\|f\|^*_{L^{p,q}})^q &= p \int_0^\infty \lambda^q |\{x : \sum_n 2^n \mathbb{1}_{F_n} > \lambda\}|^{q/p} \frac{d\lambda}{\lambda} \\
&= p \sum_{m\in\mathbb{Z}} \int_{2^{m-1}}^{2^m} \lambda^q |\{x : \sum_n 2^n \mathbb{1}_{F_n} > \lambda\}|^{q/p} \frac{d\lambda}{\lambda}
\end{aligned}$$

For $2^{m-1} \le \lambda < 2^m$, $\{x : \sum 2^n \mathbb{1}_{F_n}(x) > \lambda\} = \bigcup_{n\ge m} F_n$.

$$\begin{aligned}
&\sim \sum_{m\in\mathbb{Z}} \int_{2^{m-1}}^{2^m} \lambda^q \left(\sum_{n\ge m} |F_n| \right)^{q/p} \frac{d\lambda}{\lambda} \\
&\sim \sum_{m\in\mathbb{Z}} 2^{mq} \left(\sum_{n\ge m} |F_n| \right)^{q/p} \\
&\sim \left\| 2^m \left(\sum_{n\ge m} |F_n| \right)^{1/p} \right\|^q_{\ell^q_m}.
\end{aligned}$$

We wanted to show that $\|f\|^*_{L^{p,q}} \sim \|2^m |F_m|^{1/p}\|_{\ell^q_m}$. So we just need to show that $\left\| 2^m \left(\sum_{n \geq m} |F_n|\right)^{1/p} \right\|_{\ell^q_m} \sim \|2^m |F_m|^{1/p}\|_{\ell^q_m}$. We have the $\geq$ direction, so we just need the other inequality:

$$\begin{aligned} \left\| 2^m \left(\sum_{n \geq m} |F_n| \right)^{1/p} \right\|_{\ell^q_m} &\leq \left\| 2^m \sum_{n \geq m} |F_n|^{1/p} \right\|_{\ell^q_m} \\ &\lesssim \sum_{k \geq 0} 2^{-k} \|2^{m+k} |F_{m+k}|^{1/p}\|_{\ell^q_m} \end{aligned}$$

Now reindex the ℓ^q sum by $n = m + k$.

$$\begin{aligned} &\lesssim \sum_{k \geq 0} 2^{-k} \|2^n |F_n|^{1/p}\|_{\ell^q_n} \\ &\lesssim \|2^n |F_n|^{1/p}\|_{\ell^q_n}. \end{aligned}$$

□

4.2 Lorentz spaces are Banach spaces

Lemma 4.1. *Let $1 \leq q < \infty$, and let $S \subseteq 2^{\mathbb{Z}}$, the dyadic integers. Then*

$$\sum_{N \in S} N^q \leq \left(\sum_{N \in S} N \right)^q \leq \left(2 \sup_{N \in S} N \right)^q \leq 2^q \sum_{N \in S} N^q.$$

In other words, if we're summing dyadic series, when we take the L^q norm, it doesn't really matter whether we have the q inside or outside the sum.

Theorem 4.1. *For $1 < p < \infty$ and $1 \leq q \leq \infty$,*

$$\|f\|^*_{L^{p,q}} \sim \sup \left\{ \left| \int f(x) g(x) \, dx \right| : \|g\|^*_{L^{p',q'}} \leq 1 \right\}.$$

Thus, $\|\cdot\|^*_{L^{p,q}}$ is equivalent to a norm, with respect to which $L^{p,q}(\mathbb{R}^d)$ is a Banach space. Moreover, for $q \neq \infty$, the dual of $L^{p,q}$ is $L^{p',q'}$, under the natural pairing.

Remark 4.2. For $p = 1, q \neq 1$, there cannot be a norm equivalent to $\|\cdot\|^*_{L^{1,q}}$. Let's see this for $q = \infty$ and $d = 1$. Assume, towards a contradiction, that $\|\cdot\|^*_{L^{1,\infty}} \sim |||\cdot|||$. Let $f(x) = \sum_{n=1}^N \frac{1}{|x-n|}$ for $N \gg 1$. Then

$$\left\| \frac{1}{|x-n|} \right\|^*_{L^{1,\infty}} = \sup_{\lambda > 0} \lambda \left| \left\{ x : \frac{1}{|x-n|} > \lambda \right\} \right| = 2,$$

so

$$\sum_{n=1}^{N} \left\| \left| \frac{1}{|x-n|} \right| \right\| \sim \sum_{n=1}^{N} \left\| \frac{1}{|x-n|} \right\|_{L^{1,\infty}}^{*} = 2N.$$

Then we have

$$\|f\|_{L^{1,\infty}}^{*} = \sup_{\lambda>0} \lambda \left| \left\{ x : \sum_{n=1}^{N} \frac{1}{|x-n|} > \lambda \right\} \right|.$$

We claim that $\{x : \sum_{n=1}^{N} \frac{1}{|x-n|} > \frac{1}{10}\log N\} \supseteq [0, N]$. If $x = 0$, then $\sum 1/n > \log(N+1) \geq \frac{1}{10}\log N$. Now do the same with $x = 1, x = 2, \ldots$. The worst case scenario is when $x \approx N/2$, but the inequality holds in this case, too. So we have

$$\|f\|_{L^{1,\infty}}^{*} \geq \frac{1}{10}\log N \left| \left\{ x : \sum_{n=1}^{N} \frac{1}{|x-n|} > \frac{1}{10}\log N \right\} \right| \geq \frac{N \log N}{10}.$$

So we have shown that

$$|||f||| \sim \|f\|_{L^{1,\infty}}^{*} \geq \frac{N\log N}{10}.$$

This gives

$$N \log N \lesssim |||f||| \leq \sum_{n=1}^{N} \left|\left|\left| \frac{1}{|x-n|} \right|\right|\right| \sim N.$$

Let $N \to \infty$ to get a contradiction.

Now let's prove the theorem.

Proof. We may assume $f \geq 0$, $g \geq 0$. As both sides are positive homogeneous, we may assume that $\|f\|_{L^{p.q}}^{*} = 1$. We may assume $f = \sum 2^n \mathbb{1}_{F_n}$ and $g = \sum 2^m \mathbb{1}_{E_m}$ with F_n measurable, pairwise disjoint and E_n measurable, pairwise disjoint. Then

$$\begin{aligned} 1 &= (\|f\|_{L^{p,q}}^{*})^q \\ &\sim \|2^n |F_n|^{1/p}\|_{\ell^q}^q \\ &\sim \sum_{n\in\mathbb{Z}} 2^{nq} |F_n|^{q/p} \\ &\sim \sum_{N\in 2^{\mathbb{Z}}} \sum_{n: N \leq |F_n| < 2N} 2^{nq} |F_n|^{q/p} \\ &\sim \sum_{N\in 2^{\mathbb{Z}}} N^{q/p} \sum_{n:|F_n|\sim N} 2^{nq} \end{aligned}$$

By the lemma,

$$\sim \sum_{N\in 2^{\mathbb{Z}}} N^{q/p} \left(\sum_{n:|F_n|\sim N} 2^n \right)^q$$

$$\sim \sum_{N\in 2^{\mathbb{Z}}} \left(\sum_{n:|F_n|\sim N} 2^n |F_n|^{1/p} \right)^q .$$

Similarly,

$$1 \geq \left(\|g\|_{L^{p',q'}}\right)^{q'} \sim \sum_{M\in 2^{\mathbb{Z}}} \left(\sum_{m:|E_m|\sim M} 2^m |E_m|^{1/p'} \right)^{q'} .$$

Now

$$\begin{aligned}
\int f(x)g(x)\,dx &= \sum_{n,m} 2^n 2^m |F_n \cap E_m| \\
&\lesssim \sum_{N,M\in 2^{\mathbb{Z}}} \sum_{n:|F_n|\sim N} \sum_{m:|E_m|\sim M} 2^n |F_n|^{1/p} 2^m |E_m|^{1/p'} \frac{\min\{N,M\}}{N^{1/p}M^{1/p'}} \\
&\lesssim \sum_{N,M\in 2^{\mathbb{Z}}} \left(\frac{\min\{N,M\}}{N^{1/p}M^{1/p'}}\right)^{1/q+1/q'} \sum_{n:|F_n|\sim N} 2^n |F_n|^{1/p} \sum_{m:|E_m|\sim M} 2^m |E_m|^{1/p'}
\end{aligned}$$

By Hölder's inequality,

$$\begin{aligned}
&\lesssim \left[\sum_{N,M\in 2^{\mathbb{Z}}} \frac{\min\{N,M\}}{N^{1/p}M^{1/p'}} \left(\sum_{n:|F_n|\sim N} 2^n |F_n|^{1/p} \right)^q \right]^{1/q} \\
&\quad \cdot \left[\sum_{N,M\in 2^{\mathbb{Z}}} \frac{\min\{N,M\}}{N^{1/p}M^{1/p'}} \left(\sum_{m:|F_m|\sim M} 2^m |E_n|^{1/p'} \right)^{q'} \right]^{1/q'} .
\end{aligned}$$

Now we just need $\sum_{M\in 2^{\mathbb{Z}}} \frac{\min\{N,M\}}{N^{1/p}M^{1/p'}} \lesssim 1$. This comes from

$$\sum_M \min\left\{ \left(\frac{N}{M}\right)^{1/p'}, \left(\frac{M}{N}\right)^{1/p} \right\} \lesssim \sum_{M\leq N} \left(\frac{M}{N}\right)^{1/p} + \sum_{M>N} \left(\frac{N}{M}\right)^{1/p'} \lesssim 1,$$

as we get a geometric series.[1] □

[1] Instead of using Hölder's inequality and the subsequent steps, we could alternatively use Schur's test for convergence of series. This kind of argument will be common in this course.

5 Banach Space Properties of Lorentz Spaces

5.1 Proof of completeness, duality, and more

Theorem 5.1. *For $1 < p < \infty$ and $1 \le q \le \infty$,*

$$\|f\|^*_{L^{p,q}} \sim_{p,q} \sup\left\{\left|\int f(x)g(x)\,dx\right| : \|g\|^*_{L^{p',q'}} \le 1\right\}.$$

Thus, $\|\cdot\|^*_{L^{p,q}}$ is equivalent to a norm, with respect to which $L^{p,q}(\mathbb{R}^d)$ is a Banach space. Moreover, for $q \neq \infty$, the dual of $L^{p,q}$ is $L^{p',q'}$, under the natural pairing.

Proof. Last time, we saw that it suffices to prove the equivalence for functions of the form

$$f = \sum_{n\in\mathbb{Z}} 2^n \mathbb{1}_{F_n}$$

with F_n measurable, pairwise disjoint, and $\|f\|^*_{p,q} \sim \|2^n|F_n|^{1/p}\|_{\ell^q_n} \sim 1$. Last time, we showed that $RHS \lesssim LHS$ by testing it on $g = \sum 2^m \mathbb{1}_{E_m}$ with E_m measurable, pairwise disjoint, and $\|g\|^*_{L^{p'q'}} \sim \|2^m|E_m|^{1/p'}\|_{\ell^{q'}_m} \lesssim 1$. Let's show that $LHS \lesssim RHS$.

Compare: in the case of $L^q(\mathbb{R}^d)$, we take $g = \frac{|f|^{q-1}\operatorname{sgn} f}{\|f\|_q^{q-1}}$. Here, we take

$$g = \sum \left(2^n|F_n|^{1/p}\right)^{q-1} |F_n|^{-1/p'} \mathbb{1}_{F_n}$$

Check:

$$\begin{aligned}
\int f(x)g(x)\,dx &= \sum_n 2^n \left(2^n|F_n|^{1/p}\right)^{q-1} |F_n|^{-1/p'}|F_n| \\
&= \sum_n 2^{nq}|F_n|^{q/p} \\
&= \|2^n|F_n|^{1/p}\|^q_{\ell^q} \\
&\sim (\|f\|^*_{L^{p,q}})^q \\
&\sim 1.
\end{aligned}$$

It remains to show that $\|g\|^*_{L^{p',q'}} \lesssim 1$. By the proposition which evaluates the norm as a dyadic sum,

$$\left(\|g\|^*_{L^{p',q'}}\right)^{q'} \sim \sum_{N\in 2^{\mathbb{Z}}} N^{q'}|\{x : N \le g(x) < 2N\}|^{q'/p'}$$

Note that $\{x : g \sim N\} = \bigcup_{n \in S_N} F_n$, where $S_N = \{n \in \mathbb{Z} : 2^{n(q-1)}|F_n|^{q/p-1} \sim N\} = \{n \in \mathbb{Z} : |F_n| \sim [N2^{-n(q-1)}]^{p/(q-p)}\}$.

$$\sim \sum_{N \in 2^{\mathbb{Z}}} N^{q'} \left(\sum_{n \in S_N} |F_n| \right)^{q'/p'}$$

This is a dyadic sum, so we can pull the exponent inside (by our lemma).

$$\begin{aligned}
&\sim \sum_{N \in 2^{\mathbb{Z}}} N^{q'} \sum_{n \in S_N} |F_n|^{q'/p'} \\
&\sim \sum_{n \in \mathbb{Z}} \left(2^{n(q-1)} |F_n|^{q/p-1} \right)^{q'} |F_n|^{q'/p'}
\end{aligned}$$

Use $q' = q/(q-1)$.

$$\begin{aligned}
&\sim \sum_{n \in \mathbb{Z}} 2^{nq} |F_n|^{q'(q/p-1/p)} \\
&\sim \sum_{n \in \mathbb{Z}} 2^{nq} |F_n|^{q/p} \\
&\sim (\|f\|^*_{L^{p,q}})^q \\
&\sim 1.
\end{aligned}$$

The RHS defines a norm $|||\cdot|||$. To see that $L^{p,q}$ equiped with this norm is a Banach space, one uses the usual Riesz-Fischer argument.

Step 1: If $\|f_n\| \in L^{p,q}$ are such that $\sum |||\cdot||| < \infty$, then there exists a function $f \in L^{p,q}$ such that $f = \sum f_n$ in $|||\cdot|||$.

Step 2: For a Cauchy sequence $\{f_n\}_{n \geq 1} \subseteq L^{p,q}$, we pass to a subsequence so that $\left|\left|\left| f_{k_{n+1}} - f_{k_n} \right|\right|\right| < \frac{1}{2^n}$. So by Step 1,

$$f_{k_n} = f_{k_1} + \sum_{j=2}^{n} f_{k_j} - f_{k_{j-1}} \xrightarrow{|||\cdot|||} f.$$

For $1 \leq q < \infty$, we want to show that the dual of $L^{p,q}$ is $L^{p',q'}$. Let $\ell : L^{p,q} \to \mathbb{R}$ be a linear functional, so $|||\ell(f)||| \lesssim \|f\|^*_{L^{p,q}}$. For $f = \mathbb{1}_E$ with E of finite measure,

$$\ell(\mathbb{1}_E) \lesssim |E|^{1/p}.$$

So the measure $E \mapsto \ell(\mathbb{1}_E)$ is absolutely continuous with respect to Lebesgue measure. So there exists a $g \in L^1_{\text{loc}}$ such that

$$\ell(\mathbb{1}_E) = \int g(x) \mathbb{1}_E(x) \, dx.$$

As ℓ is linear, we get

$$\ell(f) = \int f(x)g(x)\,dx$$

for all simple functions $f \in L^{p,q}$.

- Claim 1: Boundedness of ℓ on simple functions yields $g \in L^{p',q'}$.
- Claim 2: Simple functions are dense in $L^{p,q}$ if $q \neq \infty$.

Given these two claims, we get $\ell(f) = \int f(x)g(x)\,dx$ for all $f \in L^{p,q}$. Thus, the dual of $L^{p,q}$ is $L^{p',q'}$.

Proof of Claim 1: It is enough to show that if $g = \sum 2^m \mathbb{1}_{E_m}$ with E_m measurable, pairwise disjoint, we have

$$\|2^m |E_m|^{1/p'}\|_{\ell^{q'}} \lesssim 1.$$

Choose $f = \sum_{|m|\leq M} \left(2^m |E_m|^{1/p'}\right)^{q'-1} |E_m|^{-1/p} \mathbb{1}_{E_m}$. Then

$$\ell(f) = \int f(x)g(x)\,dx \sim \|2^m |E_m|^{1/p'}\|^{q'}_{\ell^{q'}_{|m|\leq M}}, \qquad \|f\|_{L^{p,q}} \sim \|2^m |E_m|^{1/p'}\|^{q'/q}_{\ell^{q'}_{|m|\leq M}}.$$

We have $\ell(f) \lesssim \|f\|^*_{L^{p,q}}$, so

$$\|2^m |E_m|^{1/p'}\|_{\ell^{q'}_{|m|\leq M}} \lesssim \|2^m |E_m|^{1/p'}\|^{1/(q-1)}_{\ell^{q'}_{|m|\leq M}}.$$

This gives

$$\|2^m |E_m|^{1/p'}\|_{\ell^{q'}_{|m|\leq M}} \lesssim 1$$

uniformly in M. So $g \in L^{p',q'}$.

Proof of Claim 2: Consider $f \geq 0$ and look at $f_m = f\mathbb{1}_{\{2^m \leq f < 2^{m+1}\}}$. For $2^{m+n} \leq k \leq 2^{m+1+n} - 1$, let

$$E^k_{m,n} = f^{-1}\left(\left(\frac{k}{2^n}, \frac{k+1}{2^n}\right)\right), \qquad \varphi_{m,n} = \sum_{k=2}^{2^{m+n+1}-1} \frac{k}{2^n} \mathbb{1}_{E_{m,n^k}}.$$

Then $0 \leq f_m - \varphi_{m,n} \leq \frac{1}{2^n}$. First choose $\varepsilon > 0$. If we look at $\|f\|^*_{L^{p,q}} \| \|f_m\|_{L^p} \|_{\ell^q}$, only finitely many terms matter, so we can truncate the series. This lets us estimate $\|f_m - \varphi_{m,n}\|^*_{p',q'}$, as any large numbers we get will be multiplied by our small step size, $\frac{1}{2^n}$. □

6 Hunt's Interpolation Theorem

6.1 Strong type and weak type

Definition 6.1. We say that a map T on some measurable class of functions is **sublinear** if

1. $|T(cf)| \leq |c||Tf|$,
2. $|T(f+g)| \leq |T(f)| + |T(g)|$

for all constant $c \in \mathbb{C}$ and f, g in the domain of T.

Example 6.1. If T is linear, it is sublinear.

Example 6.2. If $\{T_t\}_{t\in S}$ is a family of linear maps, then

$$(Tf)(x) = \|(T_t f)(x)\|_{L_t^2}$$

is a sublinear map.

Definition 6.2. Let $1 \leq p, q \leq \infty$, and let T be a sublinear map.

1. We say that T is of **(strong) type** (p, q) if there exists a constant $C > 0$ such that
$$\|Tf\|_{L^q(\mathbb{R}^d)} \leq C\|f\|_{L^p}, \qquad \forall f \in L^p(\mathbb{R}^d).$$
2. If $q < \infty$, we say that T is of **weak-type** (p, q) if there exists a constant $C > 0$ such that
$$\|Tf\|^*_{L^{q,\infty}} \leq C\|f\|_{L^p(\mathbb{R}^d)} \qquad \forall f \in L^p(\mathbb{R}^d).$$
If $q = \infty$, we say that T is of weak-type (p, q) if it is of strong type (p, q).
3. If $p, q < \infty$, we saw that T is of **restricted weak-type** (p, q) if there exists a constant $C > 0$ such that
$$\|T\mathbb{1}_F\|^*_{L^{q,\infty}} \leq C|F|^{1/p}(\leq C\|\mathbb{1}_F\|^*_{L^{p,1}}) \qquad \forall F \subseteq \mathbb{R}^d, |F| < \infty.$$

Remark 6.1.

$$\text{Strong type } (p,q) \implies \text{weak-type } (p,q) \implies \text{restricted weak-type } (p,q).$$

For the first implication, we have $\|Tf\|^*_{L^{q,\infty}} \lesssim \|Tf\|^*_{L^{q,q}} \lesssim \|f\|_{L^p}$. For the second implication,

$$\|T\mathbb{1}_F\|^*_{L^{q,\infty}} \lesssim \|\mathbb{1}_F\|_{L^p} = \|\mathbb{1}_F\|^*_{L^{p,p}} \lesssim \|\mathbb{1}_F\|^*_{L^{p,1}} \lesssim |F|^{1/p}.$$

Exercise 6.1. For $1 < p, q < \infty$, let T be defined on functions on $(0, \infty)$ via

$$(Tf)(x) = |x|^{-1/q} \int_0^\infty |y|^{-1/p'} f(y)\, dy.$$

Then T is of restricted weak-type (p, q) but not of weak type (p, q).

Remark 6.2. Fix $1 < p, q < \infty$. If T is of restricted weak-type (p, q), then for any finite-measure sets $E, F \subseteq \mathbb{R}^d$,

$$\int |(T\mathbb{1}_F)(x)| \cdot |\mathbb{1}_E(x)|\, dx \lesssim \|T\mathbb{1}_F\|^*_{L^{q,\infty}} \|\mathbb{1}_E\|^*_{L^{q',1}} \lesssim |F|^{1/p} |E|^{1/q'}.$$

Conversely, if this condition holds for all finite measure sets $E, F \subseteq \mathbb{R}^d$, then T is of restricted weak-type (p, q). Indeed,

$$\|T\mathbb{1}_F\|^*_{L^{q,\infty}} \sim \sup_{\|g\|^*_{L^{q',1}} \leq 1} \left| \int T\mathbb{1}_F(x) g(x)\, dx \right|.$$

Take $g = \sum 2^m \mathbb{1}_{E_m}$ with E_m measurable and pairwise disjoint. Then

$$\begin{aligned}
\left| \int T\mathbb{1}_F(x) g(x)\, dx \right| &\leq \sum 2^m \int |T\mathbb{1}_F(x)| \cdot |\mathbb{1}_{E_m}(x)|\, dx \\
&\lesssim \sum 2^m |F|^{1/p} |E_m|^{1/q'} \\
&\lesssim |F|^{1/p} \|g\|^*_{L^{q',1}} \\
&\lesssim |F|^{1/p}.
\end{aligned}$$

Remark 6.3. If $1 < p, q < \infty$, then T is of restricted weak-type (p, q) if and only if there is a constant $C > 0$ such that

$$\|Tf\|^*_{L^{q,\infty}} \leq C \|f\|^*_{L^{p,1}} \qquad \forall f \in L^{p,1}(\mathbb{R}^d).$$

6.2 Hunt's interpolation theorem

Theorem 6.1 (Hunt's interpolation theorem)**.** *Let $1 \leq p_1, p_2, q_1, q_2 \leq \infty$ with $p_1 < p_2$ and $q_1 \neq q_2$. Assume that T is a sublinear map satisfying $\|Tf\|_{L^{q_j,\infty}} \lesssim \|f\|^*_{L^{p_j,1}}$ for $j = 1, 2$. Then, for any $1 \leq r \leq \infty$ and $\theta \in (0, 1)$, we have*

$$\|Tf\|^*_{L^{q_\theta,r}} \lesssim \|f\|^*_{L^{p_\theta,r}}, \qquad \frac{1}{p_\theta} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{q_\theta} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.$$

Remark 6.4.

1. If $p_\theta \leq q_\theta$, then T is of strong type (p_θ, q_θ). Indeed, taking $r = q_\theta$, we get

$$\|Tf\|_{L^{q_\theta}} \lesssim \|f\|^*_{L^{p_\theta,q_\theta}} \lesssim \|f\|_{L^{p_\theta}}.$$

2. The condition $p_\theta \le q_\theta$ is needed to obtain the strong-type conclusion. For example, let $(Tf)(x) = f(x)|x|^{-1/2}$. Then $T : L^p(0,\infty) \to L^{2p/(p+2),\infty}(0,\infty)$ boundedly for any $2 \le p < \infty$. But T is not bounded from L^p to $L^{2p/(p+2)}$ for all $2 < p < \infty$. To see that $T : L^p \to L^{2p/(p+2),\infty}$ is bounded, we use the Hölder inequality in Lorentz spaces (which we will prove later): If $1 \le p_1, p_2, p < \infty$ and $1 \le q_1, q_2, q \le \infty$, then

$$\|f_1 f_2\|^*_{L^{p,q}} \lesssim \|f_1\|^*_{L^{p_1,q_1}} \|f_1\|^*_{L^{p_2,q_2}}, \qquad \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

Then

$$\|Tf\|^*_{L^{2p/(p+2),\infty}} \lesssim \||x|^{-1/2}\|^*_{L^{2,\infty}} \|f\|^*_{L^{p,\infty}} \lesssim \|f\|_{L^p}.$$

Take

$$f(x) = |x|^{-1/p} |\log(x + 1/x)|^{-(p+2)/(2p)}.$$

We get

$$\begin{aligned} \|f\|^p_{L^p} &= \int_0^\infty |\log(x+1/x)|^{-(p+2)/(2p)} \frac{dx}{x} \\ &= \sum_{n\in\mathbb{Z}} \int_{2^n}^{2^{n+1}} |\log(x+1/x)|^{-1-p/2} \frac{dx}{x} \\ &\sim \sum_{n\in\mathbb{Z}} |n|^{-1-p/2} \\ &< \infty \end{aligned}$$

On the other hand,

$$\begin{aligned} \|Tf\|^{2p/(p+2)}_{L^{2p/(p+2)}} &= \int_0^\infty |\log(x+1/x)|^{-1} \frac{dx}{x} \\ &\sim \sum_{n\in\mathbb{Z}} |n|^{-1} \\ &= \infty \end{aligned}$$

We know that $T : L^{p_1,p_1} \to L^{2p_1/(p_1+2),\infty}$ and $T : L^{p_2,p_2} \to L^{2p_2/(p_2+2),\infty}$ for $2 \le p_1 < p_2 < \infty$. Hunt's interpolation theorem gives that for all $1 \le r \le \infty$, $T : L^{p_\theta,r} \to L^{2p_\theta/(p_\theta+2),r}$. Note that $\frac{2p_\theta}{p_\theta+2} < p_\theta$.

Next time, we will prove the following as a consequence of Hunt's interpolation theorem.

Corollary 6.1 (Marcinkiewicz interpolation theorem)**.** *Let* $1 \le p_1 \le q_1 \le \infty$ *and* $1 \le p_2 \le q_2 \le \infty$ *with* $p_1 \le p_2$ *and* $q_1 \ne q_2$*. Let* T *be a sublinear map that satisfies*

$$\|Tf\|^*_{L^{q_j,\infty}} \lesssim \|f\|_{L^{p_j}}, \qquad j = 1, 2.$$

Then for any $\theta \in (0,1)$, *T is of strong type* (p_θ, q_θ), *where*

$$\frac{1}{p_\theta} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \qquad \frac{1}{q_\theta} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.$$

We will also prove Hunt's theorem next time.

7 Proofs of Interpolation Theorems

7.1 Proof of the Marcinkiewicz interpolation theorem

Last time, we introduced Hunt's interpolation theorem.

Theorem 7.1 (Hunt's interpolation theorem)**.** *Let $1 \leq p_1, p_2, q_1, q_2 \leq \infty$ with $p_1 < p_2$ and $q_1 \neq q_2$. Assume that T is a sublinear map satisfying $\|Tf\|_{L^{q_j,\infty}} \lesssim \|f\|^*_{L^{p_j,1}}$ for $j = 1, 2$. Then, for any $1 \leq r \leq \infty$ and $\theta \in (0,1)$, we have*

$$\|Tf\|^*_{L^{q_\theta,r}} \lesssim \|f\|^*_{L^{p_\theta,r}}, \qquad \frac{1}{p_\theta} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{q_\theta} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.$$

Before proving this, we will prove the Marcinkiewicz interpolation theorem as a corollary.

Corollary 7.1 (Marcinkiewicz interpolation theorem)**.** *Let $1 \leq p_1 \leq q_1 \leq \infty$ and $1 \leq p_2 \leq q_2 \leq \infty$ with $p_1 \leq p_2$ and $q_1 \neq q_2$. Let T be a sublinear map that satisfies*

$$\|Tf\|^*_{L^{q_j,\infty}} \lesssim \|f\|_{L^{p_j}}, \qquad j = 1, 2.$$

Then for any $\theta \in (0,1)$, T is of strong type (p_θ, q_θ), where

$$\frac{1}{p_\theta} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \qquad \frac{1}{q_\theta} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.$$

Proof. As $p_1 \leq q_1$ and $p_2 \leq q_2$, we get $p_\theta \leq q_\theta$ for all $\theta \in (0,1)$. If $p_1 < p_2$, Hunt's theorem yields

$$\|Tf\|^*_{L^{q_\theta,r}} \lesssim \|f\|^*_{L^{p_\theta,r}} \qquad \forall 1 \leq r \leq \infty.$$

Taking $r = q_\theta$, we get

$$\|Tf\|_{L^{q_\theta}} \lesssim \|f\|^*_{L^{p_\theta,q_\theta}} \lesssim \|f\|_{L^{p_\theta}}.$$

Assume now that $p_1 = p_2 =: p$. Then

$$\begin{aligned}\|Tf\|^*_{L^{q_1,\infty}} \lesssim \|f\|_{L^p} &\iff \sup_{\lambda>0} \lambda |\{x : |Tf(x)| > \lambda\}|^{1/q_1} \lesssim \|f\|_p \\ &\implies |\{x : |Tf(x)| > \lambda\}| \lesssim \left(\frac{\|f\|_p}{\lambda}\right)^{q_1} \qquad \forall \lambda > 0.\end{aligned}$$

Similarly,

$$\begin{aligned}\|Tf\|^*_{L^{q_1,\infty}} \lesssim \|f\|_{L^p} &\iff \sup_{\lambda>0} \lambda |\{x : |Tf(x)| > \lambda\}|^{1/q_2} \lesssim \|f\|_p \\ &\implies |\{x : |Tf(x)| > \lambda\}| \lesssim \left(\frac{\|f\|_p}{\lambda}\right)^{q_2} \qquad \forall \lambda > 0.\end{aligned}$$

We now have

$$\begin{aligned}
\|Tf\|_{L^{q_\theta}}^{q_\theta} &= q_\theta \int_0^\infty \lambda^{q_\theta} |\{x : |(Tf)(x)| > \lambda\}| \frac{d\lambda}{\lambda} \\
&\lesssim \int_0^\infty \lambda^{q_\theta} \min\left\{ \left(\frac{\|f\|_p}{\lambda}\right)^{q_1}, \left(\frac{\|f\|_p}{\lambda}\right)^{q_2} \right\} \frac{d\lambda}{\lambda}
\end{aligned}$$

Say $q_1 < q_2$.

$$\begin{aligned}
&\lesssim \int_0^{\|f\|_p} \lambda^{q_\theta} \left(\frac{\|f\|_p}{\lambda}\right)^{q_1} \frac{d\lambda}{\lambda} + \int_{\|f\|_p}^\infty \left(\frac{\|f\|_p}{\lambda}\right)^{q_2} \frac{d\lambda}{\lambda} \\
&\lesssim \|f\|_p^{q_1} \|f\|_p^{q_\theta - q_1} + \|f\|_p^{q_2} \|f\|_p^{q_\theta - q_2} \\
&\lesssim \|f\|_p^{q_\theta}.
\end{aligned}$$

□

7.2 Proof of Hunt's interpolation theorem

Now let's prove Hunt's interpolation theorem. Recall that if $1 < p, q < \infty$, T is of restricted weak type (p, q) if

$$\|T\mathbb{1}_F\|_{L^{q,\infty}}^* \lesssim |F|^{1/p}$$

for every finite measure set F. We saw that this is equivalent to

$$\int |T\mathbb{1}_F(x)||\mathbb{1}_E(x)|\, dx \lesssim |F|^{1/p}|E|^{1/q'} \quad \forall E, F \iff \|Tf\|_{L^{q_\theta,\infty}}^* \lesssim \|f\|_{L^{p,1}}^* \quad \forall f \in L^{p,1}.$$

Proof. Claim: It suffices to prove Hunt for $1 < p_1, p_2, q_1, q_2 < \infty$. Indeed, for every $\theta \in (0, 1)$,

$$\|Tf\|_{L^{q_\theta,\infty}}^* \lesssim \|f\|_{L^{p_\theta,1}}^*$$

Indeed, for any $\theta \in (0, 1)$, even if $p_1 = 1$ and $q_1 = \infty$, $p_\theta \in (1, \infty)$. So we can use an interpolation argument with a slightly modified p_1 and p_2: It suffices to see that

$$\|T\mathbb{1}_F\|_{q_\theta,\infty}^* \lesssim |F|^{1/p_\theta}$$

for all finite measure sets F. We have

$$\begin{aligned}
\|T\mathbb{1}_F\|_{L^{q_\theta,\infty}}^* &= \sup_{\lambda > 0} \lambda^{\theta + (1-\theta)} |\{x : |Tf(x)| > \lambda\}|^{\theta/q_1 + (1-\theta)/q_2} \\
&\leq \left(\sup_{\lambda > 0} |\{x : |T\mathbb{1}_F(x)| > \lambda\}^{1/q_1}\right)^\theta \left(\sup_{\lambda > 0} |\{x : |T\mathbb{1}_F(x)| > \lambda\}^{1/q_2}\right)^{1-\theta} \\
&= \left(\|T\mathbb{1}_F\|_{L^{q_1,\theta}}^*\right)^\theta \left(\|T\mathbb{1}_F\|_{L^{q_2,\infty}}^*\right)^{1-\theta} \\
&\lesssim |F|^{1/p_1 \cdot \theta} |F|^{1/p_2 \cdot (1-\theta)} \\
&\lesssim |F|^{1/p_\theta}.
\end{aligned}$$

Henceforth, we assume $1 < p_1, p_2, q_1, q_2 < \infty$. We can write

$$\|Tf\|^*_{L^{q_\theta,r}} \sim \sup_{\|g\|^*_{L^{q'_\theta,r'}} \le 1} \left| \int Tf(x) \cdot \overline{g(x)}\, dx \right|,$$

so it's enough to show that

$$\left| \int Tf(x)\overline{g(x)}\, dx \right| \lesssim 1 \qquad \forall \|f\|^*_{L^{p_\theta,1}} = 1, \|g\|^*_{L^{q'_\theta,r'}} \lesssim 1.$$

By splitting into real and imaginary parts (and then positive and negative parts), we may assume $f, g \ge 0$. We may also assume $g = \sum 2^m \mathbb{1}_{E_m}$, where E_m are measurable and pairwise disjoint. Caution: As T need not have monotonicity properties, we may not assume f is a simple function.

Using the binary expansion, we write

$$f(x) = \sum_{n\in\mathbb{Z}} 2^n a_n(x), \qquad a_n(x) \in \{0, 1\}.$$

Note that there exists a largest $n(x)$ such that $a_{n(x)} = 1$ and $a_n(x) = 0$ for all $n > n(x)$. Also, we don't allow recurrent 1s. Let $\{n_k(x)\}_{k\ge1}$ be a decreasing sequence such that $a_{n_k(x)}(x) = 1$ and all other $a_n(x) = 0$. Then

$$f(x) = \sum_{k\ge1} 2^{n_k(x)}.$$

For $\ell \ge 1$, let $f_\ell(x) = 2^{n_\ell(x)}$. We can write

$$f_\ell(x) = \sum_{n\in\mathbb{Z}} 2^n \mathbb{1}_{F_n^\ell}, \qquad F_n^\ell = \{x : n_\ell(x) = n\}.$$

Then

$$\begin{aligned} f(x) &= 2^{n_\ell(x)} + 2^{n_{\ell-1}(x)} + \cdots + 2^{n_1(x)} \\ &\ge 2^{n_\ell(x)} + 2^{n_\ell(x)+1} + \cdots + 2^{n_\ell(x)+\ell-1} \\ &= 2^{n_\ell(x)} \cdot (2^\ell - 1) \\ &\ge f_\ell(x) \cdot 2^{\ell-1}. \end{aligned}$$

So we get

$$f_\ell(x) \le \frac{1}{2^{\ell-1}} f(x).$$

As $L^{p_\theta,r}$ is a Banach space, then $\sum_{\ell\ge1} f_\ell = f$ in $L^{p_\theta,r}$.

Now we can tackle the bound:

$$\begin{aligned}
\left|\int Tf(x)g(x)\,dx\right| &\leq \sum_{\ell\geq 1}\left|\int Tf_\ell(x)\left[\sum_m 2^m \mathbb{1}_{E_m}(x)\right]\,dx\right| \\
&\leq \sum_{\ell\geq 1}\sum_{n,m\in\mathbb{Z}} 2^n 2^m \int |T\mathbb{1}_{F_n^\ell}(x)||\mathbb{1}_{E_m}(x)|\,dx \\
&\lesssim \sum_{\ell\geq 1}\sum_{n,m} 2^n 2^m \min\{|F_n^\ell|^{1/p_1}|E_m|^{1/q_1'}, |F_n^\ell|^{1/p_2}|E_m|^{1/q_2'}\}.
\end{aligned}$$

We will show that this is $\lesssim 1$ next time. □

8 Interpolation and Maximal Function Estimates

8.1 Conclusion of proof of Hunt's interpolation theorem

Theorem 8.1 (Hunt's interpolation theorem)**.** *Let $1 \leq p_1, p_2, q_1, q_2 \leq \infty$ with $p_1 < p_2$ and $q_1 \neq q_2$. Assume that T is a sublinear map satisfying $\|Tf\|_{L^{q_j,\infty}} \lesssim \|f\|^*_{L^{p_j,1}}$ for $j = 1, 2$. Then, for any $1 \leq r \leq \infty$ and $\theta \in (0,1)$, we have*

$$\|Tf\|^*_{L^{q_\theta,r}} \lesssim \|f\|^*_{L^{p_\theta,r}}, \qquad \frac{1}{p_\theta} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{q_\theta} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.$$

Proof. We may assume $1 < p_1, p_2, q_1, q_2 < \infty$ with $p_1 < p_2$ and $q_1 \neq q_2$. We know

$$\int |T\mathbb{1}_F(x)||\mathbb{1}_E(x)|\, dx \lesssim \min\{|F_n^\ell|^{1/p_1}|E_m|^{1/q_1'}, |F_n^\ell|^{1/p_2}|E_m|^{1/q_2'}\}$$

Fix $\theta \in (0,1)$ and $1 \leq e \leq \infty$. We want to show that

$$\|Tf\|^*_{L^{q_\theta,r}} \lesssim \|f\|^*_{L^{p_\theta,r}}$$

uniformly for $f \in L^{p_\theta,r}$. It suffices to show that

$$\left|\int Tf(x)g(x)\, dx\right| \lesssim 1,$$

where $\|f\|^*_{L^{p_\theta,r}} \sim 1$ and $g = \sum_{m\in\mathbb{Z}} 2^m \mathbb{1}_{E_m}$, where E_m are measurable, pairwise disjoint, and

$$\|g\|^*_{L^{q_\theta',r'}} \sim \|2^m|E_m|^{1/q_\theta'}\|_{\ell^{r'}} \lesssim 1.$$

We write $f = \sum_{\ell\geq 1} f_\ell$, where $f_\ell = \sum_{n\in\mathbb{Z}} 2^n \mathbb{1}_{F_n^\ell}$. We have

$$\begin{aligned}\left|\int Tf(x)g(x)\, dx\right| &\lesssim \sum_{\ell\geq 1}\sum_{n,m} 2^n 2^m \min\{|F_n^\ell|^{1/p_1}|E_m|^{1/q_1'}, |F_n^\ell|^{1/p_2}|E_m|^{1/q_2'}\} \\ &\lesssim \sum_{\ell\geq 1}\sum_{n,m\in\mathbb{Z}} 2^n|F_n^\ell|^{1/p_\theta} 2^m |E_m|^{1/q_\theta'} \\ &\qquad \cdot \min\{|F_n^\ell|^{(1-\theta)(1/p_1-1/p_2)}|E_m|^{(1-\theta)(1/q_1'-1/q_2')} \\ &\qquad\qquad |F_n^\ell|^{-\theta(1/p_1-1/p_2)}|E_m|^{-\theta(1/q_1'-1/q_2')}\}\end{aligned}$$

Using the same trick we have used before, we write this as a geometric series.

$$\lesssim \sum_\ell \sum_{N,M\in 2^{\mathbb{Z}}} \sum_{n:|F_n^\ell|\sim N} 2^n N^{1/p_\theta} \sum_{m:|E_m^\ell|\sim M} 2^m M^{1/q_\theta'} A(N,M)$$

where

$$A(N,M)=\min\{|F_n^\ell|^{(1-\theta)(1/p_1-1/p_2)}|E_m|^{(1-\theta)(1/q_1'-1/q_2')},|F_n^\ell|^{-\theta(1/p_1-1/p_2)}|E_m|^{-\theta(1/q_1'-1/q_2')}\}.$$

$$\lesssim \sum_{\ell\geq 1}\left[\sum_{N,M\in 2^{\mathbb{Z}}} A(N,M)\left[\sum_{n:|F_n^\ell|\sim N} 2^n N^{1/p_\theta}\right]^r\right]^{1/r} \cdot \left[\sum_{N,M\in 2^{\mathbb{Z}}} A(N,M)\left[\sum_{m:|E_m|\sim M} 2^m M^{1/q_\theta'}\right]^r\right]^{1/r}.$$

Note that $\sup_N \sum_{M\in 2^{\mathbb{Z}}} A(N,M)\lesssim 1$ and $\sup_{M\in 2^{\mathbb{Z}}}\sum_{N\in 2^{\mathbb{Z}}} A(N,M)\lesssim 1$. Fix $M\in 2^{\mathbb{Z}}$. Let $n_0^{1/p_1-1/p_2}\sim M^{-(1/q_1'-1/q_2')}$. Then

$$\sum_{N\in 2^{\mathbb{Z}}} A(N,M)=\sum_{N\leq N_0} N^{(1-\theta)(1/p_1-1/_2)}M^{(1-\theta)(1/q_1'-1/q_2')}+\sum_{N>N_0} N^{-\theta(1/p_1-1/p_2)}M^{-\theta(1/q_1'-1/q_2')}.$$

Thus,

$$\begin{aligned}\left|\int Tf(x)g(x)\,dx\right| &\lesssim \sum_{\ell\geq 1}\left\{\sum_N\left(\sum_{n:|F_n^\ell|\sim N} 2^n N^{1/p_\theta}\right)^r\right\}^{1/r}\left\{\sum_M\left(\sum_{m:|E_m|\sim M} 2^m M^{1/q_\theta'}\right)^r\right\}^{1/r}\\ &\lesssim \sum_{\ell\geq 1}\underbrace{\left(\sum_n 2^{nr}|F_n^\ell|^{r/p_\theta}\right)^{1/r}}_{\|f_\ell\|^*_{L^{p_\theta,r}}}\underbrace{\left(\sum_m 2^{mr'}|E_m|^{r'/q_\theta'}\right)^{1/r'}}_{\|g\|^*_{L^{q_\theta',r'}}}\\ &\lesssim \sum_{\ell\geq 1}\|f_\ell\|^*_{L^{p_\theta,r}}\end{aligned}$$

Since $|f_\ell|\leq \frac{1}{2^{\ell-1}}|f|$,

$$\begin{aligned}&\lesssim \|f\|^*_{L^{p_\theta,r}}\\ &\sim 1.\end{aligned}$$

□

Remark 8.1. We did not use anything specific about Lebesgue measure in our proof. So these theorems hold for arbitrary measures μ.

8.2 Maximal and vector maximal functions

Recall the Hardy-Littlewood maximal function

$$Mf(x)=\sup_{r>0}\frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)|\,dy.$$

Theorem 8.2.

1. *If $f \in L^p(\mathbb{R}^d)$ for some $1 \le p \le \infty$, then Mf is finite almost everywhere.*

2. *M is of weak-type $(1,1)$ and strong-type (p,p) for $1 < p \le \infty$.*

Remark 8.2.

1. M is not of strong-type $(1,1)$. Let $\varphi \in C_c^\infty(B(0,1/2))$. For $|x| \le 1$, $M\varphi(x) \sim 1$. If $|x| > 1$, then $M\varphi(x) \sim \frac{1}{|x|^d}$. So $M\varphi(x) \sim \langle x \rangle^{-d}$, where this notation means $\langle x \rangle := (1 + |x|^2)^{1/2}$.[2] So $M\varphi \notin L^1$.

2. M is of weak-type $(1,1)$ means

$$|\{x : Mf(x) > \lambda\}| \lesssim \frac{\|f\|_1}{\lambda}$$

uniformly in $\lambda > 0$ and $f \in L^1$. The decay in λ on the right hand side cannot be improved. To see this, consider φ as above. Then $M\varphi \in L^\infty$, so only the small λ are relevant. For small λ,

$$\begin{aligned} |\{x : Mf(x) > \lambda\}| &= |\{x : \langle x \rangle^{-d} \gtrsim \lambda\}| \\ &= |\{x : \langle x \rangle \lesssim \lambda^{-1/d}\}| \\ &\lesssim \lambda^{-1}. \end{aligned}$$

Also, $M\varphi \notin L^{1,q}(\mathbb{R}^d)$ for any $q < \infty$ because

$$\|M\varphi\|^*_{L^{1,q}} \sim \int_0^\infty \lambda^q \underbrace{|\{x : M\varphi(x) > \lambda\}|^q}_{\lesssim \lambda^{-q}} \frac{d\lambda}{\lambda} = \infty.$$

Theorem 8.3. *Let $\omega : \mathbb{R}^d \to [0,\infty)$ be a locally integrable function (a weight), to which we associate a measure via*

$$\omega(E) = \int_E \omega(x)\,dx.$$

Then

1. *$M : L^1(M\omega\,dx) \to L^{1,\infty}(\omega\,dx)$ maps boundedly; that is,*

$$\omega(\{x : Mf(x) > \lambda\}) \lesssim \frac{1}{\lambda} \int |f(y)|(M\omega)(y)\,dy.$$

[2]This is known as "Japanese bracket notation" everywhere except Japan, where they just call it "bracket notation."

2. $M : L^p(M\omega\, dx) \to L^p(\omega\, dx)$ boundedly for all $1 < p \le \infty$; that is,

$$\int |Mf(x)|^p \omega(x)\, dx \lesssim \int |f(y)|^p (M\omega)(y)\, dy.$$

Remark 8.3.

1. If $\omega \equiv 1$, then $M\omega \equiv 1$, so we recover the previous theorem.
2. In order for the statement to be non-vacuous, we need $M\omega$ is finite somewhere. This happens precisely when $\frac{1}{r^d} \int_{|x| \le r} \omega(x)\, dx \lesssim 1$ uniformly for sufficiently large r.

   ($\implies$): If $x = 0$, we are done, so assume $x \neq 0$. For $r > 2d(x, 0)$,

   $$M\omega(x) \ge \frac{1}{|B(x,r)|} \int_{B(x,r)} \omega(y)\, dy \gtrsim \frac{1}{r^d} \int_{|x| \le r/2} \omega(y)\, dy.$$

   ($\impliedby$): Choose x to be a Lebesgue point. The Lebesgue differentiation theorem controls the maximal function at small scales, and the same argument controls the maximal function at large scales.

9 Boundedness Properties of The Hardy-Littlewood Maximal Function and A_p Weights

9.1 Boundedness properties of the Hardy-Littlewood maximal function

The Hardy-Littlewood maximal function is given by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y|\, dy.$$

Theorem 9.1. *Let $\omega : \mathbb{R}^d \to [0,\infty)$ be a locally integrable function (a weight), to which we associate a measure via*

$$\omega(E) = \int_E \omega(x)\, dx.$$

Then

1. *$M : L^1(M\omega\, dx) \to L^{1,\infty}(\omega\, dx)$ maps boundedly; that is,*

$$\omega(\{x : Mf(x) > \lambda\}) \lesssim \frac{1}{\lambda} \int |f(y)|(M\omega)(y)\, dy$$

   uniformly in $\lambda > 0$ for all $f \in L^1(M\omega\, dx)$.

2. *$M : L^p(M\omega\, dx) \to L^p(\omega\, dx)$ boundedly for all $1 < p \leq \infty$; that is,*

$$\int |Mf(x)|^p \omega(x)\, dx \lesssim \int |f(y)|^p (M\omega)(y)\, dy$$

   uniformly for $f \in L^p(M\omega\, dx)$.

Just like the proof of the maximal inequality, we will start with a covering lemma.

Lemma 9.1 (Vitali)**.** *Given a finite collection of balls $\{B(x_j, r_j)\}_{j\in J}$, there exists a subcollection S such that*

1. *Distinct balls are disjoint.*
2. *$\bigcup_{j\in I} B(x_j, r_j) \subseteq \bigcup_{j\in S} B(x_j, 3r_j)$.*

Proof. We run the following algorithm. Set $S = \varnothing$.

1. Choose a ball of largest radius and add it to S.
2. Discard any balls that intersect balls in S.
3. If no balls remain, stop. Otherwise, return to step 1. □

Now let's prove the theorem.

Proof. First note that $M : L^\infty(M\omega\, dx) \to L^\infty(\omega\, dx)$ boundedly:

$$\|Mf\|_{L^\infty(\omega\, dx)} = \inf_{E:\omega(E)=0} \sup_{x\in E^c} Mf(x)$$

Since ω is locally integrable, it takes Lebesgue-null sets to ω-null sets.

$$\begin{aligned} &\leq \inf_{E:|E|=0} \sup_{x\in E^c} Mf(x) \\ &\leq \|f\|_{L^\infty(dx)} \\ &= \inf_{E:|E|=0} \sup_{x\in E^c} |f(x)| \end{aligned}$$

$M\omega > 0$ unless $\omega \equiv 0$, so

$$\begin{aligned} &= \inf_{E:(M\omega)(E)=0} \sup_{x\in E^c} |f(x)| \\ &= \|f\|_{L^\infty(M\omega\, dx)}. \end{aligned}$$

So by the Marcinkiewicz interpolation theorem, it suffices to prove $M : L^1(M\omega\, dx)\o L^{1,\infty}(\omega\, dx)$.

Fix $\lambda > 0$. Let K be a compact subset of $\{x : Mf(x) > \lambda\}$ (this suffices by regularity). For $x \in K$, there is some $r(x) > 0$ such that

$$\frac{1}{|B(x, r(x))|} \int_{B(x,r(x))} |f(y)|\, dy > \lambda.$$

Now $K \subseteq \bigcup_{x\in K} B(x, r(x))$, and by compactness, there exists a finite subcover such that $\bigcup_{j\in J} B(x_j, r_j)$. By Vitali, there exists a subcollection S of pairwise disjoint balls such that $K \subseteq \bigcup_{j\in S} B(x_j, 3r_j)$. So $\omega(K) \leq \sum_{j\in S} \omega(B(x_j, 3r_j))$.

For Lebesgue measure, we would just pull out the constant 3 and add the measures. But here, we don't have that property, so we will relate it to the maximal function. For $x \in B(x_j, r_j)$,

$$\begin{aligned} \omega(B(x_j, 3r_j)) &= \int_{B(x_j,3r_j)} \omega(y)\, dy \\ &\leq \frac{|B(x, 4r_j)|}{|B(x, 4r_j)|} \int_{B(x,4r_j)} \omega(y)\, dy \\ &\leq 4^d |B(x, 4r_j)| M\omega(x). \end{aligned}$$

Now integrate this against f:

$$\omega(B(x_j, 3r_j)) \frac{1}{|B(x_j, r_j)|} \int_{B(x_j,r_j)} |f(y)|\, dy \leq 4^d \int_{B(x_j,r_j)} M\omega(x)|f(y)|\, dy. \qquad \square$$

Remark 9.1. Rather than placing the weights outside the maximal function, one could place them inside: Define

$$M_\mu f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)|\, d\mu(y),$$

where μ is a nonnegative measure. If μ is a **doubling measure** (i.e. if $B_1 = B(x,r)$ and $B_2 = B(x,2r)$, then $\mu(B_2) \lesssim \mu(B_1)$ uniformly for $x \in \mathbb{R}^d$ and $r > 0$), then with small modifications, the proof of this theorem yields:

$$M_\mu : L^1(d\mu) \to L^{1,\infty}(d\mu), \qquad M_\mu : L^p(d\mu) \to L^p(d\mu), \qquad \forall 1 < p \leq \infty$$

boundedly.

9.2 A_p weights

Can one characterize the nonnegative measure μ for which

$$M : L^p(d\mu) \to L^p(d\mu), \qquad 1 < p < \infty$$

boundedly? Yes, these are the A_p weights.

Definition 9.1. We say that a locally integrable weight $\omega : \mathbb{R}^d \to [0,\infty)$ satisfies the A_1 condition (and we write $\omega \in A_1$) if there is a $C > 0$ such that $M\omega(x) \leq C\omega(x)$ for almost every x.

Remark 9.2. If $\omega \in A_1$, then the theorem yields

$$M : L^p(\omega\, dx) \to L^p(\omega\, dx) \qquad M : L^1(\omega, dx) \to L^{1,\infty}(\omega\, dx) \qquad \forall 1 < p \leq \infty$$

boundedly.

Let's characterize these weights.

Lemma 9.2. *The following are equivalent:*

1. $\omega \in A_1$
2. $$\frac{1}{|B|} \int_B \omega(y)\, dy \lesssim \omega(x)$$
   uniformly for a.e. $x \in B$ and all balls B.
3. $$\frac{1}{|B|} \int_B f(y)\, dy \lesssim \frac{1}{\omega(B)} \int_B f(y)\omega(y)\, dy$$
   for all balls B and all $f \geq 0$.

Proof. (1) $\implies$ (2): Fix x with $M\omega(x) \leq C\omega(x)$, and let B be a ball of radius r that contains x. Then

$$\begin{aligned}\frac{1}{|B|}\int_B \omega(y)\,dy &\leq \frac{2^d}{|B(x,2r)|}\int_{B(x,2r)} \omega(y)\,dy \\ &\leq 2^d M\omega(x) \\ &\leq 2^d C\omega(x).\end{aligned}$$

(2) $\implies$ (3): ω is bounded below by its maximal function, so

$$\begin{aligned}\frac{1}{\omega(B)}\int_B f(y)\omega(y)\,dy &\geq \frac{1}{\omega(B)}\int_B f(y)\left(\frac{1}{|B|}\int_B \omega(z)\,dz\,dy\right) \\ &\geq \frac{1}{|B|}\int_B f(y)\,dy.\end{aligned}$$

(3) $\implies$ (2): Let x be a Lebesgue point for ω, and let $B \ni x$. Let $r \ll 1$ be such that $B(x,r) \subseteq B$. Set $f = \mathbb{1}_{B(x,r)}$. Then

$$\frac{1}{|B|}|B(x,r)| \lesssim \frac{1}{\omega(B)}\int_{B(x,r)} \omega(y)\,dy.$$

Rearranging this, we get

$$\frac{\omega(B)}{|B|} \lesssim \frac{1}{|B(x,r)|}\int_{B(x,r)} \omega(y)\,dy \to \omega(x).$$

□

Definition 9.2. We say that a weight $\omega : \mathbb{R}^d \to [0,\infty)$ satisfies the A_p condition for $1 < p < \infty$ if there exists an $A > 0$ such that

$$\sup_{\text{balls } B} \frac{1}{|B|}\int_B \omega(y)\,dy \cdot \left[\frac{1}{|B|}\int_B \omega(y)^{-p'/p}\,dy\right]^{p/p'} \leq A,$$

or equivalently,

$$\sup_{\text{balls } B} |B|^{-p}\omega(B)\|\omega^{-1/(p-1)}\|_{L^1(B)}^{p-1} \leq A.$$

Remark 9.3.

1. This condition is invariant under $\omega \mapsto \lambda\omega$ and $\omega(x) \mapsto \omega(\lambda x)$.
2. $\omega \in A_p$ if and only if $\sigma = \omega^{-p'/p} \in A_{p'}$. Indeed, the condition reads:

$$\sup_{\text{balls } B} \frac{1}{|B|^p}\int \sigma(y)^{-p/p'}\,dy\left[\int_B \sigma(y)\,dy\right]^{p/p'} \leq A.$$

   If we raise everything to the power p'/p,

$$\sup_{\text{balls } B} \frac{1}{|B|^{p'}}\int_B \sigma(y)\,dy\left[\int_B \sigma(y)^{-p/p'}\,dy\right]^{p'/p} \leq A^{p'/p}.$$

10 Characterization of A_p Weights

10.1 A_p weights for $p > 1$

Last time, we began to tackle the problem of characterizing nonnegative measures μ for which

$$\int |Mf(y)|^p \, d\mu(y) \lesssim |f(y|^p \, d\mu(p).$$

uniformly for $f \in L^p(d\mu)$ and some $1 < p < \infty$. We will not prove the full details, but we will give a compelling intuition of the results.

Fix $1 < p < \infty$. Recall that a locally integrable weight $\omega : \mathbb{R}^d \to [0, \infty)$ satisfies the A_p **condition** if there is an $A > 0$ such that

$$\sup_{\text{balls } B} \frac{1}{|B|} \int_B \omega(y) \, dy \left[\frac{1}{|B|} \int_B \omega^{-p'/p}(y) \, dy \right]^{p/p'} \leq A.$$

This is equivalent to

$$\sup_{\text{balls } B} |B|^{-p} \omega(B) \|\omega^{-1/(p-1)}\|_{L^1(\mathbb{R})}^{p-1} \leq A.$$

Remark 10.1. If $1 < p < q < \infty$, then $A_p \subseteq A_q$: Let $\omega \in A_p$. Then by Hölder,

$$\begin{aligned} \|\omega^{-1/(q-1)}\|_{L^1(B)} &\leq \|\omega^{-1/q-1}\|_{L^{(q-1)/(p-1)}} |B|^{1-(p-1)/(q-1)} \\ &= \|\omega^{-1/(p-1)}\|_{L^1(B)}^{(p-1)/(q-1)} |B|^{(q-p)/(q-1)}. \end{aligned}$$

So we get

$$|B|^{-q} \omega(B) \|\omega^{-1/(q-1)}\|_{L^1(B)}^{q-1} \leq |B|^{-p} \omega(B) \|\omega^{-1/(p-1)}\|_{L^1(B)}^{p-1} \lesssim 1$$

uniformly in B.

Lemma 10.1. *Fix $1 \leq p < \infty$. Then $\omega \in A_p$ if and only if*

$$\left[\frac{1}{|B|} \int_B f(y) \, dy \right]^p \lesssim \frac{1}{\omega(B)} \int_B |f(y)|^p \omega(y) \, dy,$$

uniformly in $f \geq 0$ and balls B.

We proved this last time for $p = 1$.

Proof. It remains to consider $1 < p < \infty$.

($\implies$):

$$\left[\frac{1}{|B|} \int_B f(y) \, dy \right]^p = \left[\frac{1}{|B|} \int_B f(y) \omega^{1/p - 1/p} \, dy \right]^p$$

$$\le |B|^{-p} \int |f(y)|^p \omega(y)\, dy \int_B \underbrace{\left[\omega(y)^{-p'/p}\, dy\right]^{p/p'}}_{\lesssim |B|^p \cdot 1/\omega(B)}$$

$$\lesssim \frac{1}{\omega(B)} \int_B |f(y)|^p \omega(y)\, dy.$$

($\Longleftarrow$): Fix $\varepsilon > 0$ and a ball B. Let $f = (\omega+\varepsilon)^{-p'/p}$. Then

$$\begin{aligned}\left[\frac{1}{|B|}\int_B (\omega+\varepsilon)^{-p'/p}(y)\, dy\right]^p &\lesssim \frac{1}{\omega(B)} \int_B (\omega+\varepsilon)^{-p'}(y)\omega(y)\, dy \\ &\lesssim \frac{1}{\omega(B)} \int (\omega+\varepsilon)^{-p'+1}(y)\, dy\end{aligned}$$

Note that $p'-1 = -p'/p$.

$$\lesssim \frac{1}{\omega(B)} \int_B (\omega+\varepsilon)^{-p'/p}(y)\, dy.$$

So

$$|B|^{-p}\omega(B)\left[\int_B (\omega+\varepsilon)^{-p'/p}(y)\, dy\right]^{p/p'} \lesssim 1$$

uniformly in B and $\varepsilon > 0$. Let $\varepsilon \to 0$ and use the monotone convergence theorem □

Corollary 10.1. *Fix $1 \le p < \infty$. If $\omega \in A_p$, then ω is a doubling measure.*

Proof. Let $B = B(x, 2r)$ and $f = \mathbb{1}_{B(x,r)}$. Then

$$\left[\frac{|B(x,r)|}{|B(x,2r)|}\right]^p \lesssim \frac{\omega(B(x,r))}{\omega(B(x,2r))}$$

uniformly in $x \in \mathbb{R}^d$ and $r > 0$. □

Remark 10.2. In fact, A_p weights with $1 \le p < \infty$ satisfy a "fairness condition": If $F \subseteq B$, then taking $f = \mathbb{1}_F$, we get

$$\left[\frac{|F|}{|B|}\right]^p \lesssim \frac{\omega(F)}{\omega(F)}.$$

So if F is a large chunk of the ball B, ω has to give a large proportion of the measure of the ball to F; it has to treat F fairly.

10.2 Proof sketch of characterization

Theorem 10.1. *Fix $1 < p < \infty$. Then $\omega \in A_p$ if and only if*

$$\int |Mf(y)|^p \omega(y)\,dx \lesssim \int |f(y)|^p \omega(y)\,dy$$

uniformly for $f \in L^p(\omega\,dx)$ (that is, $M : L^p(\omega\,dx) \to L^p(\omega\,dx)$ boundedly).

This answers the question we proposed but only in the case that ω is a weight; i.e. ω is absolutely continuous with respect to Lebesgue measure.

Remark 10.3. ($\Longleftarrow$): The necessity holds under even weaker assumptions. If $M : L^p(\omega\,dx) \to L^{p,\infty}(\omega\,dx)$ boundedly, then $\omega \in A_p$.

Proof. ($\Longleftarrow$): Fix a ball B of radius $r > 0$. Fix $\varepsilon > 0$, and let $f = (\omega + \varepsilon)^{-p'/p} \mathbb{1}_B$. For $x \in B$,

$$\begin{aligned} Mf(x) &= \sup_{R>0} \frac{1}{|B(x,R)|} \int_{B(x,R)} (\omega + \varepsilon)^{-p'/p} \mathbb{1}_B(x)\,dx \\ &\geq \frac{1}{|B(x,2r)|} \int_B (\omega + \varepsilon)^{-p'/p}(y)\,dy \\ &= \frac{1}{2^d |B|} \int_B (\omega + \varepsilon)^{p'/p}(y)\,dy \end{aligned}$$

Let's give this a name

$$=: 2\lambda.$$

Now

$$\begin{aligned} \omega(B) &\leq \omega(\{x : Mf(x) > \lambda\}) \\ &\lesssim \frac{\int_B (\omega + \varepsilon)^{-p'}(y) \omega(y)\,dy}{\lambda^p} \\ &\lesssim \frac{\int_B (\omega + \varepsilon)^{-p'/p}(y)\,dy}{[\int_B (\omega + \varepsilon)^{-p'/p}(y)\,dy]^p}. \end{aligned}$$

So we get that

$$|B|^{-p} \omega(B) \left[\int_B (\omega + \varepsilon)^{p'/p}(y)\,dy \right]^{p/p'} \lesssim 1.$$

□

Remark 10.4. ($\Longrightarrow$): The sufficiency, that is, $\omega \in A_p \implies M$ is bounded on $L^p(\omega\,dx)$, rests on three ingredients:

1. If $1 \leq q < \infty$, then $M : L^q(\omega\, dx) \to L^{q,\infty}(\omega\, dx)$ boundedly (this is Homework exercise 10). Look at
$$M_\omega f(x) = \sup_{r>0} \frac{1}{\omega(B(x,r))} \int_{B(x,r)} |f(y)|\omega(y)\, dy.$$
Then $M_\omega : L^1(\omega\, dx) \to L^{1,\infty}(\omega\, dx)$ boundedly. Then
$$\left|\frac{1}{|B|} \int_B f(y)\, dy\right|^p \lesssim \frac{1}{\omega(B)} \int_B |f(y)|^p \omega(y)\, dy,$$
which tells us
$$|Mf|^p \lesssim M_\omega(|f|^p).$$

2. (Appears in Ch5 of Stein's Harmonic Analysis textbook[3]) A reverse Hölder inequality: If $\omega \in A_\infty = \bigcup_{1 \leq p < \infty} A_p$, then there exist and $r > 1$ and $c > 0$ (both depending on ω) such that
$$\left[\frac{1}{|B|} \int_B \omega^r(y)\, dy\right]^{1/r} \leq \frac{c}{|B|} \int_B \omega(y)\, dy \iff |B|^{1/r'} \|\omega\|_{L^r(B)} \leq c\|\omega\|_{L^1(B)}.$$
This implies that if $\omega \in A_p$, then $\omega \in A_q$ for some $q < p$.

Ingredients 1 and 2 give $M : L^q(\omega\, dx) \to L^{q,\infty}(\omega\, dx)$ boundedly for some $q < p$.

3. The Marcinkiewicz interpolation theorem with $M : L^\infty(\omega\, dx) \to L^\infty(\omega\, dx)$ (use the fact that $|E| = 0 \iff \omega(E) = 0$ since $\omega > 0$ a.e. as $\omega \in A_p$).

Next time, we will discuss how this generalizes to arbitrary measures, not just ones absolutely continuous with respect to Lebesgue measure.

[3] This book is the bible of Harmonic Analysis.

11 A_p Weights and The Vector-Valued Maximal Function

11.1 Use of reverse Hölder in the characterization of A_p weights

Last time, we proved the following theorem:

Theorem 11.1. *Fix $1 < p < \infty$. Then $\omega \in A_p$ if and only if $M : L^p(\omega\, dx) \to L^p(\omega\, dx)$ boundedly.*

We showed that ($\Longleftarrow$) holds if $L : L^p(\omega\, dx) \to L^{p,\infty}(\omega\, dx)$. For the ($\Longrightarrow$) direction, we had 3 ingredients:

1. $M : L^1(\omega\, dx) \to L^{q,\infty}(\omega, dx)$ for all $1 \leq q < \infty$.
2. A reverse Hölder inequality yields if $\omega \in A_p$, then $\omega \in A_q$ for some $q < p$.
3. $M : L^\infty(\omega,\, dx) L^\infty(\omega\, dx)$ boundedly.

The reverse holds inequality says

Lemma 11.1. *If $\omega \in A_p$, then there exist an $r > 1$ and $c > 0$ such that*

$$\left(\frac{1}{|B|}\int_B \omega(y)^r\, dy\right)^{1/r} \leq \frac{c}{|B|}\int_B \omega(y)\, dy.$$

We will not prove this. Here's how we use it:

Proof. Apply this to $\sigma(y) = \omega(y)$. Recall that $\omega \in A_p \iff \sigma \in A_p$. Then there exist $r > 1$ and $c > 0$ depending on σ (and hence on ω) so that

$$\left[\frac{1}{|B|}\int_B \omega(y)^{-rp'/p}\, dy\right]^{1/r} \leq \frac{C}{|B|}\int_B \omega(y)^{-p'/p}\, dy.$$

So we get

$$\begin{aligned}\omega \in A_p &\iff \sup_B \frac{1}{|B|}\omega(B)\left(\frac{1}{|B|}\int_B \omega(y)^{-p'/p}\, dy\right)^{p/p'} \lesssim 1 \\ &\implies \frac{1}{|B|}\int_B \omega(y)^{-p'/p}\, dy \lesssim \left(\frac{|B|}{\omega(B)}\right)^{p'/p} \lesssim \left(\frac{|B|}{\omega(B)}\right)^{1/(p-1)}.\end{aligned}$$

We get

$$|B|^{-1/r}\left(\int_B \omega(y)^{-rp'/p}\, dy\right)^{1/r} \lesssim \left(\frac{|B|}{\omega(B)}\right)^{1/(p-1)}.$$

Write

$$rp'/p = q'/q \iff r\frac{1}{p-1} = \frac{1}{q-1}$$

$$\iff a - 1 = \frac{p-1}{p} < p - 1$$
$$\iff q = 1 + \frac{p-1}{p} \in (1, p).$$

We get

$$|B|^{-q/q' \cdot 1/(p-1)} \left(\int_B \omega(y)^{-q'/q} \, dy \right)^{q/q' \cdot 1/(p-1)} \lesssim \left(\frac{|B|}{\omega(B)} \right)^{1/(p-1)}$$
$$|B|^{-1-(q-1)} \omega(B) \left(\int_B \omega(y)^{-q'/q} \, dy \right)^{q/q'} \lesssim 1.$$

So $\omega \in A_q$. □

Theorem 11.2. *Fix $1 \le p < \infty$. If $d\mu$ is a nonnegative Borel measure such that $M : L^p(d\mu) \to L^{p,\infty}(d\mu)$ boundedly, then $d\mu = \omega \, dx$ and $\omega \in A_p$.*

Proof. It suffices to show that $d\mu$ is absolutely continuous with respect to Lebesgue measure. Write $d\mu = \omega \, dx + d\nu$ with $d\nu$ singular with respect to Lebsegue measure. Let K be a compact set such that $|K| = 0$ and $\nu(K) > 0$. For $n \ge 1$, let $U_n = \{: d(x,k) < 1/n\}$. Note that $U_n \setminus K \supseteq U_{n+1} \setminus K$ and $\bigcap (U_n \setminus K) = \varnothing$. Let $f_n = \mathbb{1}_{U_n \setminus K}$, so $f_{n+1} \le f_n$ and $f_n \to 0$.

We claim that $d\mu$ is finite on compact sets. Assuming the claim, by the monotone convergence theorem,

$$\int |f_n|^p \, d\mu \xrightarrow{n \to \infty} 0.$$

For $x \in K$,

$$\begin{aligned} M f_n(x) &= \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} \mathbb{1}_{U_n \setminus K}(y) \, dy \\ &\ge \frac{1}{|B(x,1/n)|} \int_{B(x,1/n)} \mathbb{1}_{U_n \setminus K}(y) \, dy \\ &= \frac{1}{|B(x,1/n)|} \int_{K^c} \mathbb{1}_{B(x,1/n)}(y) \, dy \end{aligned}$$

As $|K| = 0$,

$$\begin{aligned} &= \frac{1}{|B(x,1/n)|} \int_{\mathbb{R}^d} \mathbb{1}_{B(x,1/n)}(y) \, dy \\ &= 1 \end{aligned}$$

Then

$$\mu(K) \le \mu(\{x : M f_n(x) > 1/2\}) \lesssim \int |f_n|^p \, d\mu \xrightarrow{n \to \infty} 0,$$

so we get a contradiction.

Now we prove the claim. Let E be a compact set such that $0 < \mu(E) < \infty$.

$$\begin{aligned} M\mathbb{1}_E(x) &= \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} \mathbb{1}_E(y)\, dy \\ &\gtrsim \frac{|E|}{[d(x,E) + \operatorname{diam}(E)]^d} \end{aligned}$$

So $M\mathbb{1}_E$ is bounded from below uniformly on compact sets: if F is a compact set, then for $x \in F$,

$$M\mathbb{1}_E(x) \lesssim \frac{|E|}{[\operatorname{dist}(F,E) + \operatorname{diam} F + \operatorname{diam} E]^d} =: C(F,E).$$

Then

$$\mu(F) \le \mu\left(\left\{x : M\mathbb{1}_E(x) > \frac{1}{2}C(F,E)\right\}\right) \lesssim_{F,E} \int |\mathbb{1}_E(y)|^p \, d\mu(y) \lesssim_{F,E} \mu(E) < \infty. \quad \square$$

11.2 The vector-valued maximal function

Definition 11.1. Let $F : \mathbb{R}^d \to \ell^2$, $f(x) = \{f_n(x)\}_{n\ge1}$. We write

$$|f(x)| = \|\{f_n(x)\}_{n\ge1}\|_{\ell^2}, \qquad \|f\|_{L^p} = \left(\int |f(x)|^p \, dx\right)^{1/p}.$$

The **vector-valued maximal function** is

$$\overline{M}f(x) = \|\{Mf_n(x)\}_{n\ge1}\|_{\ell^2}.$$

Theorem 11.3.

1. *$\overline{M}$ is of weak-type $(1,1)$.*
2. *For $1 < p < \infty$, $\overline{M}$ is of strong type (p,p).*

Remark 11.1. We no longer have a trivial L^∞ bound. In fact, it fails. Take $d = 1$. For $n \ge 1$, take $f_n = \mathbb{1}_{[2^{n-1},2^n)}$.

$$|f(x)| = \sqrt{\sum_{n\ge1} |f_n|^2(x)} = \mathbb{1}_{[1,\infty)}(x) \in L^\infty$$

For $|x| \le 2^n$,

$$Mf_n(x) = \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} \mathbb{1}_{[2^{n-1},2^n)}(y)\, dy$$

$$
\begin{aligned}
&\geq \frac{1}{2 \cdot 2^{n+1}} \int_{x-2^{n+1}}^{x+2^{n+1}} \mathbb{1}_{[2^{n-1}, 2^n)}(y)\, dy \\
&= \frac{1}{2 \cdot 2^{n+1}} 2^{n-1} \\
&= \frac{1}{8}.
\end{aligned}
$$

Now

$$
\begin{aligned}
\overline{M} f(x) &= \sqrt{\sum_{n \geq 1} |M f_n(x)|^2} \\
&\geq \sqrt{\sum_{n : 2^n \geq |x|} \left(\frac{1}{8}\right)^2} \\
&= \infty
\end{aligned}
$$

So $\overline{M} f \notin L^\infty$.

Remark 11.2. Boundedness of $\overline{M}$ on L^2 follows from the scalar case:

$$
\begin{aligned}
\|\overline{M} f\|_{L^2}^2 = \int \sum_{n \geq 1} |M f_n(x)|^2 &= \sum_{n \geq 1} \|M f_n\|_{L^2}^2 \lesssim \sum_{n \geq 1} \|f_n\|_{L^2}^2 \\
&= \sum_{n \geq 1} \int |f_n(x)|^2\, dx \leq \int |f(x)|^2\, dx = \|f\|_{L^2}^2.
\end{aligned}
$$

Let's prove boundedness of $\overline{M}$ on L^p for $2 < p < \infty$.

Proof. If $\omega \geq 0$ with $\omega \in L^1_{\text{loc}}$, then

$$
\int |M f_n|^2 \omega\, dx \lesssim \int |f_n|^2 (M\omega)\, dx
$$

uniformly in n. Summing in n, we get

$$
\int |\overline{M} f(x)|^2 \omega(x)\, dx \lesssim \int |f(x)|^2 (M\omega)(x)\, dx.
$$

Then

$$
\begin{aligned}
\|\overline{M} f\|_{L^p}^2 &= \||\overline{M} f|^2\|_{L^{p/2}} \\
&= \sup_{\|\omega\|_{L^{(p/2)'}} \leq 1} \int |\overline{M} f|^2(x) \omega(x)\, dx
\end{aligned}
$$

$$
\begin{aligned}
&\lesssim \sup_{\|\omega\|_{L^{(p/2)'}}\leq 1} \int \underbrace{|f(x)|^2}_{\in L^{p/2}} \underbrace{(M\omega)(x)}_{\in L^{(p/2)'}}\, dx \\
&\lesssim \||f|^2\|_{L^{p/2}} \sup_{\|\omega\|_{L^{(p/2)'}}\leq 1} \underbrace{\|M\omega\|_{L^{(p/2)'}}}_{\lesssim \|\omega\|_{L^{(p/2)'}}} \\
&\lesssim \|f\|_{L^p}^2.
\end{aligned}
$$

To prove M is of strong-type (p,p) for $1 < p < \infty$, it suffices (by Marcinkiewicz) to show that $\overline{M}$ is of weak-type $(1,1)$. □

We will use the following.

Lemma 11.2 (A Calderón-Zygmund decomposition)**.** *If $f \in L^1(\mathbb{R}^d)$ and $\lambda > 0$, then we can decompose $f = g + b$ such that*

1. $|g(x)| \leq \lambda$ *for almost every $x \in \mathbb{R}^d$.*
2. $\operatorname{supp} b$ *is a union of cubes whose interiors are pairwise disjoint and*

$$\lambda < \frac{1}{|Q_k|}\int_{Q_k} |b(x)|\, dx \leq 2^d \lambda.$$

3. $g = f[1 - \mathbb{1}_{\bigcup Q_k}]$.

Next time, we will prove this decomposition and use it to prove the weak $(1,1)$ bound.

12 Calderón-Zygmund Decomposition and Bounds for the Vector-Valued Maximal Function

12.1 A Calderón-Zygmund decomposition

Lemma 12.1 (A Calderón-Zygmund decomposition). *If $f \in L^1(\mathbb{R}^d)$ and $\lambda > 0$, then we can decompose $f = g + b$ such that*

1. $|g(x)| \le \lambda$ *for almost every* $x \in \mathbb{R}^d$.
2. $\operatorname{supp} b$ *is a union of cubes whose interiors are pairwise disjoint and*
$$\lambda < \frac{1}{|Q_k|} \int_{Q_k} |b(x)|\, dx \le 2^d \lambda.$$
3. $g = f[1 - \mathbb{1}_{\bigcup Q_k}]$.

Remark 12.1.
1. Interpolating between the first conclusion and $g \in L^1$, we get $g \in L^p$ for all $1 \le p \le \infty$.
2. $\sum_k |Q_k| \sim \frac{1}{\lambda} \sum_k \int_k \int_{Q_k} |b(y)|\, dy$, so $\sum_k |Q_k| \lesssim \frac{1}{\lambda} \|f\|_{L^1}$.

Remark 12.2. Modifying g further, we can ensure that $\int_{Q_k} b(y)\, dy = 0$ for all k. Indeed, let
$$g(x) = \begin{cases} f(x) & x \notin \bigcup_k Q_k \\ \frac{1}{|Q_k|} \int_{Q_k} f(y)\, dy & x \in Q_k. \end{cases}$$
Then for $x \in Q_k$,
$$b(x) = f(x) - \frac{1}{|Q_k|} \int_{Q_k} f(y)\, dy,$$
so
$$\int_{Q_k} b(x)\, dx = \int_{Q_k} f(x)\, dx - \int_{Q_k} f(y)\, dy = 0.$$
We lose a factor of 2 for the constant:
$$\frac{1}{|Q_k|} \int_{Q_k} |b(x)|\, dx \le \frac{2}{|Q_k|} \int_{Q_k} |f(x)|\, dx \le 2^{d+1} \lambda.$$
The price we have to pay is that $|g(x)| \le 2^d \lambda$ instead of λ.

Proof. Decompose $\mathbb{R}^d$ into dyadic cubes $Q = [2^n k_1, 2^n(k_1 + 1)] \times \cdots \times [2^n k_d, 2^n(k_d + 1)]$, where n is sufficiently large so that
$$\frac{1}{|Q|} \int_Q |f(y)|\, dy \le \lambda$$
Fix such a Q and subdivide it into 2^d congruent cubes (cut each side in half). Let Q' denote one of the resulting children.

- If $\frac{1}{|Q'|}\int_{Q'}|f(y)|\,dy > \lambda$, stop and add Q' to the collection Q_k.
- If $\frac{1}{|Q'|}\int_{Q'}|f(y)|\,dy \leq \lambda$, then continue subdividing until (if ever) we are forced into case 1.

If we are in case 1, then

$$\lambda < \frac{1}{|Q'|}\int_{Q'}|f(y)|\,dy \leq \frac{2^d}{|Q|}\int_{Q}|f(y)|\,dy \leq 2^d\lambda.$$

It remains to show that $g = f[1-\mathbb{1}_{\bigcup Q_k}]$ satisfies $|g| \leq \lambda$ a.e. Fix a Lebesgue point $x \notin \bigcup Q_k$ for f. Then

$$\left|\frac{1}{|Q|}\int_Q f(y)\ dy - f(x)\right| \leq \frac{1}{|Q|}\int_Q |f(y)-f(x)|\,dx$$

for any cube, we can inscribe a ball in side it and we can circumscribe a ball around it. Letting $r \sim \operatorname{diam}(Q)$,

$$\begin{aligned} &\lesssim \frac{1}{|B(x,r)|}\int_{B(x,r)}|f(y)-f(x)\,dx \\ &\xrightarrow{r\to\infty} 0 \end{aligned}$$

So

$$f(x) = \lim_{\substack{x\ni Q \\ \operatorname{diam} Q\to 0}} \frac{1}{|Q|}\int_Q f(y)\,dy,$$

and we get $|g(x)| = |f(x)| \leq \lambda$. □

12.2 Weak-type bound for the vector-valued maximal function

Recall that for $f : \mathbb{R}^d\ to \ell^2$ with $f = \{f_n\}_{n\geq 1}$, the **vector-valued maximal function** is

$$\overline{M}f(x) = \|\{Mf_n\}_{n\geq 1}\|_{\ell^2}.$$

Theorem 12.1.

1. *$\overline{M}$ is of weak-type $(1,1)$.*
2. *For $1 < p < \infty$, $\overline{M}$ is of strong type (p,p).*

Proof. Last time, we remarked that we need only prove part 1. Fix $f \in L^1$ and $\lambda > 0$. We want to show that

$$|\{x : \overline{M}f(x) > \lambda\}| \lesssim \frac{\|f\|_{L^1}}{\lambda}.$$

Decompose $f = g + b$ with $|g| \le \lambda$ a.e., $\operatorname{supp} b = \bigcup_k Q_k$, and $\frac{1}{|Q_k|}\int_{Q_k} f|b(y)|\,dy \sim \lambda$. Then

$$|\{x : \overline{M}f(x) > \lambda\}| \le |\{x : \overline{M}g(x) > \lambda/2\}| + |\{x : \overline{M}b(x) > \lambda/2\}|.$$

By Chebyshev,

$$|\{x : \overline{M}g(x) > \lambda/2\}| \lesssim \frac{\|\overline{M}g\|_2^2}{\lambda^2} \lesssim \frac{\|g\|_2^2}{\lambda^2} \lesssim \frac{\lambda\|g\|_{L^1}}{\lambda^2} \lesssim \frac{\|f\|_{L^1}}{\lambda}.$$

It is left to show that

$$|\{x : \overline{M}b(x) > \lambda/2\}| \lesssim \frac{\|f\|_{L^1}}{\lambda}.$$

We have

$$\sum_k |2Q_k| \le 2^d \sum_k |Q_k| \sim \sum_k \frac{1}{\lambda}\int_{Q_k} |b(y)|\,dy \lesssim \frac{\|f\|_{L^1}}{\lambda}.$$

We have to show now that

$$|\{x \in [\bigcup(2Q_k)]^c : \overline{M}b(x) > \lambda/2\}| \lesssim \frac{\|f\|_{L^1}}{\lambda}.$$

For $x \notin \bigcup(2Q_k)$,

$$\begin{aligned} Mb_n(x) &= \sup_{r>0} \frac{1}{|B(x,r)|}\int_{B(x,r)} |b_n(y)|\,dy \\ &= \sup_{r>0} \frac{1}{|B(x,r)|}\sum_k \int_{B(x,r)\cap Q_k} |b_n(y)|\,dy \end{aligned}$$

If $B(x,r)\cap Q_k \ne \varnothing$, then $r > \ell(Q_k)/2$. So $Q_k \subseteq B(x, r + \sqrt{d}\ell(Q_k)) \subseteq B(x, r(1+2\sqrt{d}))$.

$$\begin{aligned} &\le_d \sup_{r>0} \frac{1}{|B(x,r(1+2\sqrt{d}))|}\sum_k \int_{B(x,r(1+2\sqrt{d}))} |b_n(y)|\,dy \\ &\lesssim \sup_{r>0} \frac{1}{|B(x,r(1+2\sqrt{d}))|}\int_{B(x,r(1+2\sqrt{d}))} \sum_k \mathbb{1}_{Q_k}(z)\left(\frac{1}{|Q_k|}\int_{Q_k} |b_n(y)|\,dy\right)\,dz. \end{aligned}$$

Let $b_n^{\text{avg}} = \sum \mathbb{1}_{Q_k}\frac{1}{|Q_k|}\int_{Q_k} |b_n(y)|\,dy$. Then we have

$$Mb_n(x) \lesssim Mb_n^{\text{avg}}(x).$$

Let $b^{\text{avg}} = \{b_n^{\text{avg}}\}_{n\ge1}$. For $x \in Q_k$,

$$|b^{\text{avg}}(x)| = \|\{b_n^{\text{avg}}(x)\}_n\|_{\ell^2} \le \frac{1}{|Q_k|}\int_{Q_k} |b(y)|\,dy \lesssim \lambda.$$

We also have

$$\|b^{\text{avg}}\|_{L^1} = \sum_k \int_{Q_k} |b(y)|\, dy \lesssim \|f\|_{L^1}.$$

By Chebyshev, since $\overline{M}b(x) \lesssim Mb^{\text{avg}}(x)$,

$$\begin{aligned}
|\{x \in [\bigcup(2Q_k)]^c : \overline{M}b(x) > \lambda/2\}| &\lesssim |\{x \in [\bigcup(2Q_k)]^c : \overline{M}b^{\text{avg}} \gtrsim\}| \\
&\lesssim \frac{1}{\lambda^2}\|\overline{M}b^{\text{avg}}\|\|_{L^2}^2 \\
&\lesssim \frac{\|b^{\text{avg}}\|_{L^2}^2}{\lambda^2} \\
&\lesssim \frac{\|b^{\text{avg}}\|_{L^1}}{\lambda} \lesssim \frac{\|f\|_{L^1}}{\lambda}.
\end{aligned}$$

□

Remark 12.3. One can replace ℓ^2 by ℓ^q for $1 < q \leq \infty$ for $f : \mathbb{R}^d\ toell^q$. Define

$$\overline{M}_q f(x) = \|\{Mf_n(x)\}_{n\geq 1}\|_{\ell^q}.$$

Then

1. $\overline{M}_q$ is of weak-type $(1,1)$.
2. $\overline{M}_q$ is of strong type (p,p) for all $1 < p < \infty$.

The proof is as in the case $q = 2$ if $1 < q < \infty$. The trivial estimate becomes that $\overline{M}_q : L^q \to L^q$ is bounded. If $q = \infty$,

$$\overline{M}_\infty f \leq M\|\{f_n\}_{n\geq 1}\|_{\ell^\infty}.$$

The estimates follow from the scalar case.

If $q = 1$, then these estimates fail. We will see an example next time.

13 The Hardy-Littlewood-Sobolev Inequality

13.1 Failure of bounds for L^1 vector-valued maximal function

Let $f : \mathbb{R}^d \to \ell^1$, $f = \{f_n\}_{n\geq 1}$ with $|f(x)| = \sum_{n\geq 1} |f_n(x)|$. We define the L^1 vector-valued maximal function as $\overline{M}_1 f(x) = \sum_{n\geq 1} M f_n(x)$. The the following claims fail:

1. $|\{x : \overline{M}_1 f(x) > \lambda\}| \leq \frac{1}{\lambda}\|f\|_{L^1}$ uniformly for $\lambda > 0, f \in L^1$.
2. For $1 < p < \infty$, $\|\overline{M}_1 f\|_{L^p} \lesssim \|f\|_{L^p}$ uniformly for $f \in L^p$.

Fix $d = 1$. Take $[0, 1]$ and subdibide it into intervals $I_1, \ldots, I_N$ of equal length. Let

$$f_n = \begin{cases} \mathbb{1}_{I_n} & 1 \leq n \leq N \\ 0 & n > N. \end{cases}$$

Let $f = \{f_n\}_{n\geq 1}$. Then

$$|f(x)| = \sum_{n\geq 1} |f_n(x)| = \mathbb{1}_{[0,1]}(x) \in L^p \qquad \forall 1 \leq p \leq \infty,$$

$$\|f\|_{L^p} = 1 \qquad \forall 1 \leq p \leq \infty.$$

On the other hand,

$$\begin{aligned} \overline{M}_1 f(x) &= \sum_{n=1}^{N} M f_n(x) \\ &= \sum_{n=1}^{N} \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} \mathbb{1}_{I_n}(y)\, dy \end{aligned}$$

For $x \in [0, 1]$, $\overline{M} f(x) \geq \sum_{n=1}^{\lfloor N/2 \rfloor} \frac{1}{2(n/N)} \cdot \frac{1}{N} \gtrsim \log(N)$. This tells us that

$$|\{x : \overline{M}_1 f(x) > \frac{1}{10} \log N\}| \geq 1,$$

$$\|\overline{M}_1 f\|_{L^p} \gtrsim \log N.$$

13.2 The Hardy-Littlewood-Sobolev inequality

Theorem 13.1 (Hardy-Littlewood-Sobolev)**.** *Fix $1 < p < r < \infty$ and $1 < q < \infty$ such that $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Then*

$$\|f * g\|_{L^r} \lesssim \|f\|_p \|g\|^*_{L^{q,\infty}},$$

uniformly for $f \in L^p, g \in L^{q,\infty}$*. In particular, for* $0 < \alpha < d$*,*

$$\left\| f * \frac{1}{|x|^\alpha} \right\|_{L^r} \lesssim \|f\|_{L^p},$$

provided $1 + \frac{1}{r} = \frac{1}{p} + \frac{\alpha}{d}$.

Proof. Fix $g \in L^{q,\infty}$. We may assume that $\|g\|^*_{L^{q,\infty}} = 1$. We want to show that the sublinear operator $f \overset{T}{\mapsto} f * g$ is of strong type (p, r). By the Marcinkiewicz interpolation theorem, it suffices to show T is of weak type (p, r) for all $1 < p < r < \infty$ such that $1 + 1/r = 1/p + 1/q$. Say the target is strong-type (p_0, r_0). Then choose $1 < p_1 < p_0 < p_2 < \infty$ and write $\frac{1}{p_0} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$. By Marcinkiewicz, if T is of weak-type (p_1, r_1) and (p_2, r_2), then T is of strong-type $(p_0, \widetilde{r})$, where

$$\frac{1}{\widetilde{r}} = \frac{\theta}{r_1} + \frac{1-\theta}{r_2} = \theta\left[\frac{1}{p_1} + \frac{1}{q} + -1\right] + (1-\theta)\left[\frac{1}{p_2} + \frac{1}{q} - 1\right] = \frac{1}{p_0} + \frac{1}{q} + 1 = \frac{1}{r_0}.$$

Let's show T is of weak-type (p, r):

$$|\{x : |(f * g)(x)| > \lambda\}| \lesssim \left(\frac{\|f\|_p}{\lambda}\right)^r$$

We may rescale so that $\|f\|_p = 1$. Write $g = g_1 + g_2 = g\mathbb{1}_{\{|g| \le R\}} + g\mathbb{1}_{\{|g|>R\}}$. Then

$$|\{x : |(f * g)(x)| > \lambda\}| \lesssim |\{x : |(f * g_1)(x)| > \lambda/2\}| + |\{x : |(f * g_2)(x)| > \lambda/2\}|$$

By Chebyshev,

$$\begin{aligned}
|\{x : |(f * g_1)(x)| > \lambda/2\}| &\lesssim \frac{\|f * g_1\|_s^s}{\lambda^s} \\
&\lesssim \frac{\|f\|_p^s \|g_1\|^s_{ps/(ps+p-s)}}{\lambda^s} \\
&\lesssim \lambda^{-s}\left(\int_0^\infty \alpha^{ps/(ps+p-s)} \mathbb{1}_{\{|g_1|>\alpha\}} \frac{d\alpha}{\alpha}\right)^{(ps+p-s)/p} \\
&= \lambda^{-s}\left(\int_0^R \alpha^{ps/(ps+p-s)} \mathbb{1}_{\{|g|>\alpha\}} \frac{d\alpha}{\alpha}\right)^{(ps+p-s)/p} \\
&= \lambda^{-s}(\|g\|^*_{L^{q,\infty}})^{q\cdot(ps+p-s)/p}\left(\int_0^R \alpha^{ps/(ps+p-s)-q} \frac{d\alpha}{\alpha}\right)^{(ps+p-s)/p} \\
&\lesssim \lambda^{-s} R^{s-q\cdot(ps+p-s)/p},
\end{aligned}$$

Provided $\frac{1}{q} > 1 + \frac{1}{s} - \frac{1}{p}$. Since $1 + \frac{1}{r} = \frac{1}{q} + \frac{1}{p}$, this means that $s > r$.

On the other hand, by Chebyshev,

$$\begin{aligned}
|\{x : |(f * g_2)(x)| > \lambda/2\}| &\lesssim \frac{\|f * g_2\|_p^p}{\lambda^p} \\
&\lesssim \frac{\|f\|_p^p \|g_2\|_1^p}{\lambda^p} \\
&\lesssim \lambda^{-p} \left(\int_0^\infty |\{x : |g_2|(x) > \alpha\}| \, d\alpha \right)^p \\
&\lesssim \lambda^{-p} \left(\|g\|_{L^{q,\infty}}^*\right)^{pq} \left(\int_R^\infty \alpha^{-q} \, dx \right)^p \\
&\lesssim \lambda^{-p} R^{(1-q)p}.
\end{aligned}$$

Optimize in R:

$$\begin{aligned}
\lambda^{-s} R^{s-qs(1+1/s-1/p)} &= \lambda^{-p} R^{(1-q)p} \\
\lambda^{p-s} &= R^{(1-q)(p-s)} R^{q/p\cdot(p-s)} \\
\lambda = R^{1-q+q/p} &= R^{q(1/q+1/p-1)} = R^{q/r}.
\end{aligned}$$

So we optimize at $R = \lambda^{r/q}$.

So

$$\begin{aligned}
|\{x : |f * g|(x) > \lambda\}| &\lesssim \lambda^{-p} \lambda^{r/q\cdot p(1-q)} \\
&\lesssim \lambda^{-p(1-r/q+r)} \\
&\lesssim \lambda^{-pr(1/r-1/q+1)} \\
&\lesssim \lambda^{-pr/p} \\
&\lesssim \lambda^{-r}.
\end{aligned}$$

□

Although we have just proven this claim, here is Hedberg's proof of $\|f * \frac{1}{|x|^\alpha}\|_r \lesssim \|f\|_p$ whenever $0 < \alpha < d$ and $1 + \frac{1}{r} = \frac{1}{p} + \frac{\alpha}{d}$.

Proof. Fix $x \in \mathbb{R}^d$. Then

$$\left(f * \frac{1}{|x|^\alpha} \right)(x) = \int \frac{f(y)}{|x-y|^\alpha} \, dy = \int_{|x-y|\leq R} \frac{f(y)}{|x-y|^\alpha} \, dy + \int_{|x-y|>R} \frac{f(y)}{|x-y|^\alpha} \, dy.$$

$$\begin{aligned}
\left| \int_{|x-y|\leq R} \frac{f(y)}{|x-y|^\alpha} \, dy \right| &\leq \sum_{\substack{r\in 2^{\mathbb{Z}} \\ r\leq R}} \int_{R\leq|x-y|\leq 2r} \frac{f(y)}{|x-y|^\alpha} \, dy \\
&\lesssim \sum_{r\leq R} r^{-\alpha} r^d \frac{1}{|B(x,2r)|} int_{B(x,2r)} |f(y)| \, dy
\end{aligned}$$

$$\lesssim R^{d-\alpha} Mf(x).$$

On the other hand,

$$\begin{aligned}
\left| \int_{|x-y|>R} \frac{f(y)}{|x-y|^\alpha}\, dy \right| &= \left| f * \frac{\mathbb{1}_{\{|x|>R\}}}{|x|^\alpha} \right| (x) \\
&\lesssim \|f\|_p \left\| \frac{\mathbb{1}_{\{|x|>R\}}}{|x|^\alpha} \right\|_{p'} \\
&\lesssim \|f\|_p \int_R^\infty \frac{r^{d-1}}{r^{\alpha p'}}\, dr \\
&\lesssim \|f\|_p R^{d/p'-\alpha} \\
&\lesssim \|f\|_p R^{d(1-1/p-\alpha/d)} \\
&\lesssim \|f\|_p R^{-d/r}.
\end{aligned}$$

Optimize in R: choose

$$R^{d-\alpha} Mf(x) = \|f\|_p R^{-d/r}$$

$$R^{d/p} = R^{d(1-\alpha/d+1/r)} = \frac{\|f\|_p}{Mf(x)}.$$

So

$$\left| \left(f * \frac{1}{|x|^\alpha} \right)(x) \right| \lesssim \|f\|_p \left(\frac{\|f\|_p}{Mf(x)} \right)^{-p/r} \lesssim Mf(x)^{p/r} \|f\|_p^{1-p/r}.$$

So

$$\begin{aligned}
\left\| f * \frac{1}{|x|^\alpha} \right\|_r &\lesssim \|f\|_p^{1-p/r} \|(Mf)^{p/r}\|_r \\
&\lesssim \|f\|_p^{1-p/r} \|Mf\|_p^{p/r} \\
&\lesssim \|f\|_p.
\end{aligned}$$

□

14 The Sobolev Embedding Theorem

14.1 Fourier transforms of tempered distributions

Fix $0 < \alpha < d$, and consider

$$\int e^{-\pi t|x|^2} t^{(d-\alpha)/2} \frac{dt}{t}.$$

If we let $u = \pi |x|^2 t$, then this equals

$$\begin{aligned} \int_0^\infty e^{-u} \left(\frac{u}{\pi |x|^2} \right)^{(d-\alpha)/2} \frac{du}{u} &= \pi^{(d-\alpha)/2} \frac{1}{|x|^{d-\alpha}} \int_0^\infty e^{-u} u^{(d-\alpha)/2} \frac{du}{u} \\ &= \pi^{-(d-\alpha)/2} \Gamma\left(\frac{d-\alpha}{2} \right) \frac{1}{|x|^{d-\alpha}}. \end{aligned}$$

We regard $\pi^{-(d-\alpha)/2}\Gamma\left(\frac{d-\alpha}{2}\right)\frac{1}{|x|^{d-\alpha}}$ as a **tempered distribution**, that is an element of $\mathcal{S}'(\mathbb{R}^d)$. These are linear functionals on $\mathcal{S}(\mathbb{R}^d)$.

For $T \in \mathcal{S}'(\mathbb{R}^d)$givenby a density φ,,

$$T(f) = \int f(x)\varphi(x)\, dx.$$

In our case,

$$\left(\pi^{-(d-\alpha)/2} \Gamma\left(\frac{d-\alpha}{2} \right) \frac{1}{|x|^{d-\alpha}} \right)(f) = \pi^{-(d-\alpha)/2} \Gamma\left(\frac{d-\alpha}{2} \right) \int \frac{f(x)}{|x|^{d-\alpha}}\, dx.$$

Since f is a Schwarz function, this integrand has the right decay at ∞. We have

$$\begin{aligned} \left| \int \frac{f(x)}{|x|^{d-\alpha}}\, dx \right| &\leq \left| \int_{|x| \leq R} \frac{f(x)}{|x|^{d-\alpha}}\, dx \right| + \left| \int_{|x|>R} \frac{f(x)}{|x|^{d-\alpha}}\, dx \right| \\ &\lesssim \|f\|_\infty \int \frac{1}{|x|^{d-\alpha}}\, dx + \||x|^d f\|_{L^\infty} R^{-(d-\alpha)}. \end{aligned}$$

Definition 14.1. For $T \in \mathcal{S}'(\mathbb{R}^d)$, we define its **Fourier transform** by

$$\widehat{T}(f) = T(\widehat{f}), \qquad f \in \mathcal{S}(\mathbb{R}^d).$$

Let's compute the Fourier transform of $\pi^{-(d-\alpha)/2}\Gamma\left(\frac{d-\alpha}{2}\right)\frac{1}{|x|^{d-\alpha}}$:

$$\begin{aligned} \left(\pi^{-(d-\alpha)/2} \Gamma\left(\frac{d-\alpha}{2} \right) \frac{1}{|x|^{d-\alpha}} \right)^{\wedge} (f) &= \pi^{-(d-\alpha)/2} \Gamma\left(\frac{d-\alpha}{2} \right) \int \frac{\widehat{f}(x)}{|x|^{d-\alpha}}\, dx \\ &= \pi^{-(d-\alpha)/2} \Gamma\left(\frac{d-\alpha}{2} \right) \iint \frac{e^{-2\pi i x \cdot \xi}}{|x|^{d-\alpha}} f(\xi)\, dx\, d\xi \end{aligned}$$

$$= \int_0^\infty \iint e^{-\pi t|x|^2} t^{(d-\alpha)/2} e^{-2\pi i x\cdot\xi} f(\xi)\,dx\,d\xi\,\frac{dt}{t}$$

We already know the Fourier transform of a Gaussian.

$$= \int_0^\infty \int \pi^{d/2}(\pi t)^{-d/2} e^{-\pi/t\cdot|\xi|^2} t^{(d-\alpha)/2} f(\xi)\,d\xi\,\frac{dt}{t}$$
$$= \int_0^\infty \int e^{-\pi/t\cdot|\xi|^2} t^{-\alpha/2} f(\xi)\,d\xi\,\frac{dt}{t}$$

Make the change of variables $u = \pi|\xi|^2/t$.

$$= \int \int_0^\infty f(\xi) e^{-u} \left(\frac{u}{\pi|\xi|^2}\right)^{\alpha/2} \frac{du}{u}\,d\xi$$
$$= \int \pi^{-\alpha/2} \frac{1}{|\xi|^\alpha} \int_0^\infty e^{-u} u^{\alpha/2} \frac{du}{u} f(\xi)\,d\xi$$
$$= \left[\pi^{-\alpha/2}\Gamma(\alpha/2)\frac{1}{|\xi|^\alpha}\right](f)$$

Remark 14.1. Take $d = 3$ and $\alpha = 2$:

$$\left(\pi^{-1/2}\Gamma(1/2)\frac{1}{|x|}\right)^\wedge = \pi^{-1}\Gamma(1)\frac{1}{|\xi|^2}.$$

That is,

$$\left(\frac{1}{2\pi|x|}\right)^\wedge = \frac{1}{4\pi^2|\xi|^2}.$$

This allows us to solve Poisson's equation: $-\Delta u = f$. If we take the Fourier transform, this is

$$4\pi^2|\xi|^2 \widehat{u}(\xi) = \widehat{f}(\xi),$$

so

$$\widehat{u}(\xi) = \frac{1}{4\pi^2|\xi|^2}\widehat{f}(\xi).$$

Taking the inverse Fourier transform, we get

$$u = \frac{1}{4\pi|x|} * f.$$

To make everything rigorous, use

- If $T \in \mathcal{S}'(\mathbb{R}^d)$ and $f \in \mathcal{S}(\mathbb{R}^d)$, then $T * f \in \mathcal{S}'(\mathbb{R}^d)$ is given by $(T * f)(g) = T(f_R * g)$, where $f_R(x) = f(-x)$:

$$(T * f)(g) = \int (\varphi * f)(x) g(x)\,dx$$

$$
\begin{aligned}
&= \iint \varphi(y) f(x-y) g(x)\, dx\, dy \\
&= T(f_R * g).
\end{aligned}
$$

- If $T \in \mathcal{S}'(\mathbb{R}^d)$ and $f \in \mathcal{S}(\mathbb{R}^d)$, then $\widehat{T * f} = \widehat{T}\widehat{f}$.

14.2 Sobolev embedding

Definition 14.2. Fix $s > -d$ and $f \in \mathcal{S}(\mathbb{R}^d)$. Then $|\nabla|^s f \in \mathcal{S}'(\mathbb{R}^d)$ is defined by its action on the Fourier side:

$$(|\nabla|^s f)^\wedge(\xi) = (2\pi|\xi|)^s \widehat{f}(\xi).$$

Theorem 14.1 (Sobolev embedding)**.** *For $f \in \mathcal{S}(\mathbb{R}^d)$ and $0 < s < d$, we have*

$$\|f\|_q \lesssim \||\nabla|^s f\|_p$$

whenever $\frac{1}{p} = \frac{1}{q} + \frac{s}{d}$. The implicit constant is independent of f.

What does this say? It says that if s derivatives of f live in L^p, then the function must be more regular/smooth (it lives is a higher L^p space).

Proof. By duality, $\|f\|_q = \sup_{\|g\|_{L^{q'}}=1} \langle f, g\rangle$. The idea is that by Plancherel,

$$\langle f, g\rangle = \langle \widehat{f}, \widehat{g}\rangle = \langle (2\pi|\xi|)^s \widehat{f}, (2\pi|\xi|^{-s}\widehat{g}(\xi)\rangle,$$

where the first argument is in $\mathcal{S}'$.

We claim that $\mathcal{F} = \{g \in \mathcal{S}'(\mathbb{R}^d) : \widehat{g} \text{ vanishes on a neighborhood of } 0\}$ is dense in $L^{q'}$. It suffices to show that $\mathcal{F}$ is dense in $\mathcal{S}(\mathbb{R}^d)$ in the topology of $L^{q'}$. Fix $g_0 \in \mathcal{S}(\mathbb{R}^d)$. Fix $\varepsilon > 0$ and $\varphi \in C_c^\infty(B(0,2))$ with $\varphi \equiv 1$ on $B(0,1)$. Define $\widehat{g_\varepsilon(\xi)} = \widehat{g_0(\xi)}(1 - \varphi(\xi/\varepsilon)) \in \mathcal{S}$. Then $g_\varepsilon \in \mathcal{F}$. Then $\widehat{g}_0 - \widehat{g}_\varepsilon = \widehat{g}_0 \varphi(\cdot/\varepsilon)$, so $g_0 - g_\varepsilon = g_0 * \varepsilon^d \varphi^\vee(\varepsilon\, \cdot)$.

$$
\begin{aligned}
\|g_0 - g_\varepsilon\|_{q'} &\lesssim \|g_0\|_1 \|\varepsilon^d \varphi^\vee(\varepsilon\, \cdot)\|_{q'} \\
&\lesssim \|g_0\|_1 \varepsilon^{d - d/q'} \\
&\xrightarrow{\varepsilon \to 0} 0.
\end{aligned}
$$

Then

$$
\begin{aligned}
\|f\|_q &= \sup_{g \in \mathcal{F} : \|g\|_{q'}=1} \langle f, g\rangle \\
&= \sup_{g \in \mathcal{F} : \|g\|_{q'}=1} \langle \underbrace{(2\pi|\xi)^s \widehat{f}}_{\in \mathcal{S}'}, \underbrace{(2\pi|\xi|)^{-s}\widehat{g}}_{\in \mathcal{S}} \rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \sup_{g\in\mathcal{F}:\|g\|_{q'}=1} \langle \underbrace{|\nabla|^s f}_{\in\mathcal{S}'}, \underbrace{|\nabla^{-s} g}_{\in\mathcal{S}} \rangle \\
&\lesssim \sup_{g\in\mathcal{F}:\|g\|_{q'}=1} \||\nabla|^s f\|_p \cdot \||\nabla|^{-s} g\|_{p'}
\end{aligned}
$$

We have

$$
|\nabla|^{-s} g = [(2\pi|\xi|)-s\widehat{g}]^{\vee} \sim \frac{1}{|x|^{d-s}} * g,
$$

so

$$
\||\nabla|^{-s} g\|_{p'} \sim \left\| \frac{1}{|x|^{d-s} * g} \right\|_{p'} \lesssim \|g\|_{q'}
$$

by Hardy-Littlewood-Sobolev, provided $1 + \frac{1}{p'} = \frac{1}{q'} + \frac{d-s}{d}$. We can rewrite this condition as $\frac{1}{p} = \frac{1}{q} = \frac{s}{d}$. $\square$

15 Riesz Tranforms and Calderón-Zygmund Convolution Kernels

15.1 Riesz tranforms

Last time, we proved the Sobolev embedding theorem:

Theorem 15.1 (Sobolev embedding)**.** *For $f \in \mathcal{S}(\mathbb{R}^d)$ and $0 < s < d$, we have*

$$\|f\|_q \lesssim \||\nabla|^s f\|_p$$

whenever $\frac{1}{p} = \frac{1}{q} + \frac{s}{d}$. The implicit constant is independent of f.

In particular,

$$\|f\|_q \lesssim \||\nabla| f\|_p, \qquad \frac{1}{p} = \frac{1}{q} + \frac{1}{d}.$$

However, the Fourier transform is not a local operator; it looks at the whole function. However, we can ask whether it is true that

$$\|f\|_q \lesssim \|\nabla f\|_p \qquad \frac{1}{p} = \frac{1}{q} + \frac{1}{d}$$

with $1 < p < q < \infty$. This would follow from boundedness of the Riesz transforms on L^p for $1 < p < \infty$.

Definition 15.1. For $1 \leq j \leq d$ and $f \in \mathcal{S}(\mathbb{R}^d)$, we define the **Riesz transforms** as

$$\widehat{R_j f}(\xi) = m_j(\xi)\widehat{f}(\xi) = -\frac{i\xi_j}{|\xi|}\widehat{f}(\xi).$$

In other words, $R_j = -\frac{\partial_j}{|\nabla|}$.

We write

$$2\pi|\xi| = \sum_{j=1}^d m_j(\xi) \cdot 2\pi i \xi_j.$$

So

$$|\nabla| = \sum_{j=1}^d R_j \partial_j.$$

Then

$$\|f\|_q \lesssim \||\nabla|\|_p \leq \sum_{j=1}^d \|R_j \partial_j f\| \lesssim \sum_{j=1}^d \|\partial_j f\|_p \lesssim \|\nabla f\|_p,$$

if we knew the Riesz transforms were bounded on L^p. (The last step comes from the fact that all finite dimensional vector space norms are equivalent.)

Remark 15.1. If we knew that the Riesz transforms are bounded on L^p for $1 < p < \infty$, we could also conclude that the solution u to the Poisson equation $-\Delta u = f$ satisfies $\partial_j \partial_k u \in L^p$ whenever $f \in L^p$. Indeed,

$$(\partial_j \partial_k u)^\wedge(\xi) = -4\pi^2 \xi_j \xi_k \widehat{u}(\xi) = -\frac{\xi_j \xi_k}{|\xi|^2} \widehat{f}(\xi) = m_j(\xi) m_k(\xi) \widehat{f}(\xi).$$

So $\partial_j \partial_k u = R_j R_k f$.

How do we prove boundedness of Riesz transforms?

Definition 15.2. A function $K : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ is a **Calderón-Zygmund convolution kernel** if it satisfies:

1. $|K(x)| \lesssim |x|^{-d}$ uniformly for $|x| > 0$.
2. $\int_{R_1 \leq |x| \leq R_2} L(x)\, dx = 0$ for all $0 < R_1 < R_2 < \infty$ (cancellation condition).
3. $\int_{|x| > 2|y|} |K(x+y) - K(x)|\, dx \lesssim 1$ uniformly for $y \in \mathbb{R}^d$ (regularity condition).

Example 15.1. The Riesz tranforms correspond to Calderón-Zygmund convolution kernels.

$$m_j = -\frac{i\xi_j}{|\xi|} \implies k_j(x) = m_j^\vee(x) = -\frac{1}{2\pi} \partial_j \left[\frac{\pi^{-(d-1)/2} \Gamma((d-1)/2)}{\pi^{-1/2} \Gamma(1/2)} \right]^{(d-1/2)} \sim_d \frac{x_j}{|x|^{d+1}}.$$

We have

1. $|k_j(x)| \lesssim x^{-d}$ uniformly in $|x| > 0$.
2. $\int_{R_1 \leq |x| \leq R_2} k_j(x)\, dx = 0$ for all $0 < R_1 < R_2 < \infty$ because it is odd in x_j.
3. By the fundamental theorem of calculus,

$$\int_{|x| \geq 2|y|} |k_j(x+y) - k_j(x)|\, dx \leq \int_{|x| > 2|y|} |y| \int_0^1 |\nabla k_j(x + \theta y)|\, d\theta\, dx$$

   For $|x| > 2y|$ and $\theta \in (0,1)$, $|x|/2 \leq |x| - |y| \leq |x + \theta y| \leq |x| + |y| \leq 3|x|/2$.

$$\lesssim \int_{|x| > 2|y|} |y| \frac{1}{|x|^{d+1}}\, dx$$
$$\lesssim |y| \cdot \frac{1}{|y|} \lesssim 1.$$

More generally, we have proved the following.

Lemma 15.1. *If $|\nabla K(x)| \lesssim |x|^{-(d+1)}$ uniformly for $|x| > 0$, then K satisfies the regularity condition.*

15.2 L^2-boundedness of convolution with Calderón-Zygmund convolution kernels

Here is a lemma we need.

Lemma 15.2. *Let $K : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ be a Calderón-Zygmund convolution kernel. For $\varepsilon > 0$, let $K_\varepsilon = K\mathbb{1}_{\{\varepsilon \le |x| \le 1/\varepsilon\}}$. Then K_ε is a Calderón-Zygmund convolution kernel.*

Proof. $|k_\varepsilon(x)| \lesssim |k(x)| \lesssim |x|^{-d}$ uniformly for $|x| > 0$. For the second condition,

$$\int_{R_1 \le |x| \le R_2} K_\varepsilon(x)\, dx = \int_{\max\{R_1,\varepsilon\} \le |x| \le \min\{R_2, 1/\varepsilon\}} K(x)\, dx = 0, \qquad \forall 0 < R_1 < R_2 < \infty.$$

For the third condition,

$$\begin{aligned} \int_{|x|>2|y|} |K_\varepsilon(x+y) - K_\varepsilon(y)|\, dx \le & \int_{\substack{\varepsilon \le |x| \le 1/\varepsilon \\ \varepsilon \le |x+y| \le 1/\varepsilon \\ |x|>2|y|}} |K_\varepsilon(x+y) - K_\varepsilon(y)|\, dx \\ & + \int_{\substack{\varepsilon \le |x| \le 1/\varepsilon \\ \varepsilon > |x+y| \text{ or } |x+y| > 1/\varepsilon \\ |x|>2|y|}} |K_\varepsilon(x)|\, dx \\ & + \int_{\substack{\varepsilon > |x| \text{ or } x > 1/\varepsilon \\ \varepsilon \le |x+y| \le 1/\varepsilon \\ |x|>2|y|}} |K_\varepsilon(x+y)|\, dx. \end{aligned}$$

Look at I: If $|x+y| < \varepsilon$, then $|x| \le |x+y| + |y| \le |x+y| + |x|/2$, so $|x| \le 2|x+y| \le 2\varepsilon$. The contribution is at most

$$\int_{\varepsilon \le |x| \le 2\varepsilon} |K(x)|\, dx \lesssim \int_{\varepsilon \le |x| \le 2\varepsilon} |x|^{-d}\, dx \lesssim 1,$$

uniformly in $\varepsilon > 0$.

If $|x+y| > 1/\varepsilon$, then $|x| \ge |x+y| - |y| \ge |x+y| - |x|/2$, so $|x| \ge \frac{2}{3}|x+y| \ge \frac{2}{3\varepsilon}$. The contribution is at most

$$\int_{\frac{2}{3\varepsilon} \le |x| \le \frac{1}{\varepsilon}} |K(X)|\, dx \lesssim 1,$$

uniformly in $\varepsilon > 0$. Similarly, $\mathrm{II} \lesssim 1$ uniformly in $\varepsilon > 0$ and $y \in \mathbb{R}^d$. □

Theorem 15.2. *Ket $K : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ be a Calderón-Zygmund convolution kernel. For $\varepsilon > 0$, let $K_\varepsilon = K\mathbb{1}_{\{\varepsilon < |x| \le 1/\varepsilon\}}$. THen*

$$\|K_\varepsilon * f\|_2 \lesssim \|f\|_2$$

uniformly for $\varepsilon > 0, f \in L^2$. Consequently, $f \mapsto K * f$ (which is the L^2 limit as $\varepsilon \to 0$ of $K_\varepsilon * f$) extends continuously from $\mathcal{S}(\mathbb{R}^d)$ to a bounded map on $L^2(\mathbb{R}^d)$.

Proof.

$$\begin{aligned}\|K_\varepsilon * f\|_2 &= \|\widehat{K_\varepsilon * f}\|_2 \\ &= \|\widehat{K}_\varepsilon \widehat{f}\|_2 \\ &\leq \|\widehat{K}_\varepsilon\|_\infty \|whf\|_2 \\ &\leq \|\widehat{K}_\varepsilon\|_\infty \|f\|_2.\end{aligned}$$

Fix $\xi \in \mathbb{R}^d$. Then

$$\begin{aligned}\widehat{K}_\varepsilon(\xi) &= \int e^{-2\pi i x\cdot\xi} K_\varepsilon(x)\,dx \\ &= \int_{|x|\leq 1/|\xi|} e^{-2\pi i x\cdot\xi} K_\varepsilon(x)\,dx + \int_{|x|>1/|\xi|} e^{-2\pi i x\cdot\xi} K_\varepsilon(x)\,dx.\end{aligned}$$

Now observe that by condition 2 of the definition of the Calderón-Zygmund convolution kernel,

$$\left|\int_{\varepsilon\leq|x|\leq 1/|\xi|} e^{-2\pi i x\cdot\xi} K_\varepsilon(x)\,dx\right| = \left|\int_{|x|\leq 1/|\xi|} (e^{-2\pi i x\cdot\xi} - 1) K_\varepsilon(x)\,dx\right|$$

By condition 1,

$$\begin{aligned}&\lesssim \int_{|x|\leq 1/|\xi|} |x||\xi||x|^{-d}\,dx \\ &\lesssim |\xi| \frac{1}{|\xi|} \\ &\lesssim 1.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\int_{|x|>1/|\xi|} e^{-2\pi i x\cdot\xi} K_\varepsilon(x)\,dx &= \int_{|x|>1/|\xi|} \frac{1}{2}(1 - e^{\pi i}) e^{-2\ piix\cdot\xi} K_\varepsilon(x)\,dx \\ &= \int_{|x|>1/|\xi|} \frac{1}{2} e^{-2\pi i x\cdot\xi} K_\varepsilon(x)\,dx \\ &\quad - \frac{1}{2}\int_{|x|>1/|\xi|} e^{-2\pi i \xi(x-\xi/(2|\xi|^2))} K_\varepsilon(x)\,dx \\ &= \int_{|x|>1/|\xi|} \frac{1}{2} e^{-2\pi i x\cdot\xi} K_\varepsilon(x)\,dx \\ &\quad - \frac{1}{2}\int_{\left|x+\frac{\xi}{2|\xi|^2}\right|>\frac{1}{|\xi|}} e^{-2\pi i x\cdot\xi} K_\varepsilon\left(x + \frac{|xi|}{2|\xi|^2}\right)\,dx,\end{aligned}$$

which puts us into a position to make a change of variables and use condition 3. We will finish the proof next time. □

16 Boundedness of Calderón-Zygmund Convolution Kernels

16.1 L^2-boundedness of convolution with Calderón-Zygmund kernels

Last time, we were proving the following theorem.

Theorem 16.1. *Let $K : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ be a Calderón-Zygmund convolution kernel. For $\varepsilon > 0$, let $K_\varepsilon = K \mathbb{1}_{\{\varepsilon < |x| \le 1/\varepsilon\}}$. Then*

$$\|K_\varepsilon * f\|_2 \lesssim \|f\|_2$$

uniformly for $\varepsilon > 0, f \in L^2$. Consequently, $f \mapsto K * f$ (which is the L^2 limit as $\varepsilon \to 0$ of $K_\varepsilon * f$) extends continuously from $\mathcal{S}(\mathbb{R}^d)$ to a bounded map on $L^2(\mathbb{R}^d)$.

Proof. By Plancherel,

$$\|K_\varepsilon * f\|_{L^2} \le \|\widehat{K}_\varepsilon\|_\infty \|f\|_{L^2},$$

so it suffices to show that $\|\widehat{K}_\varepsilon\|_\infty \lesssim 1$ uniformly in $\varepsilon > 0$. Fix $\xi \in \mathbb{R}^d$. Then

$$\widehat{K}_\varepsilon(\xi) = \int e^{-2\pi i x \cdot \xi} K_\varepsilon(x)\, dx$$

$$= \int_{|x| \le 1/|\xi|} + \int_{|x| > 1/|\xi|}$$

Because of property (b) and (a),

$$\begin{aligned}\int_{\varepsilon \le |x| \le 1/|\xi|} e^{-2\pi i x \cdot \xi} K_\varepsilon(x)\, dx &= \int_{|x| \le 1/|\xi||} \left| [e^{-2\pi i x \cdot \xi} - 1] K_\varepsilon(x) \right| \\ &\lesssim \int_{|x| \le 1/|\xi|} |x| \cdot |\xi| \cdot \frac{1}{|x|^d}\, dx \lesssim 1.\end{aligned}$$

We also have

$$\begin{aligned}\int_{|x| > 1/|\xi|} e^{-2\pi i x \cdot \xi} K_\varepsilon(x)\, dx &= \int_{|x| > 1/|\xi|} \frac{1}{2}(e^{-2\pi i x \cdot \xi} - e^{-2\pi i \xi \cdot (x - \xi/(2|\xi|^2))}) K_\varepsilon(x)\, dx \\ &= \frac{1}{2} \int_{|x| > 1/|\xi|} e^{-2\pi i x \cdot \xi} K_\varepsilon(x)\, dx \\ &\quad - \frac{1}{2} \int_{x + \xi/(2|\xi|^2) > 1/|\xi|} e^{-2\pi i x \cdot \xi} K_\varepsilon(x + \xi/(2\|\xi|^2))\, dx \\ &= \frac{1}{2} \int_{|x| > 1/|\xi|} e^{-2\pi i x \cdot \xi} [K_\varepsilon(x) - K_\varepsilon(x + \xi/(2|\xi|^2))]\, dx \\ &\quad + \frac{1}{2} \int e^{-2\pi i x \cdot \xi} K_\varepsilon(x + \xi/(2|\xi|^2))\, dx\end{aligned}$$

We can split $\int_{x+\xi/(2|\xi|^2)>1/|\xi|} = \int_{|x|>1/|\xi|} - \int_A - \int_B$, where A and B are a partition of the symmetric difference (like a Venn diagram). So $A = \{x : |x| \leq 1/|\xi| \leq |x + \xi/(2|\xi|^2)|\}$ and $B = \{x : |x + \xi/(2|\xi|^2)| \leq 1/|\xi| \leq |x|\}$.

$$= \underbrace{\int_{|x|>1/|\xi|} e^{-2\pi i x\cdot\xi}[K_\varepsilon(x) - K_\varepsilon(x + \xi/(2|\xi|^2))]\, dx}_{I}$$
$$- \underbrace{\frac{1}{2}\int_A e^{-2\pi i x\cdot\xi} K_\varepsilon(x + \xi/(2|\xi|^2))\, dx}_{II}$$
$$- \underbrace{\frac{1}{2}\int_B e^{-2\pi i x\cdot\xi} K_\varepsilon(x + \xi/(2|\xi|^2))\, dx}_{III}.$$

Looking at these terms individually:

$$|I| \leq \frac{1}{2}\int_{|x|\geq 1/|\xi|} |K_\varepsilon(x) - K_\varepsilon(x + \xi/(2|\xi|^2))|\, dx \lesssim 1$$

uniformly in ξ and $\varepsilon > 0$ by condition (c).

$$|II| \lesssim \int_A |K_\varepsilon(x + \xi/(2|\xi|^2))|\, dx$$

Note that $A \subseteq x : 1/|\xi| \leq |x + \xi|/(2|\xi|^2)| \leq |x| + 1/(2|xi|) \leq 3/(2|\xi|)$.

$$\lesssim \int_{1/|\xi|\leq|y|\leq 3/(2|\xi|)} |K_\varepsilon(y)|\, dy$$
$$\lesssim \int_{1/|\xi|\leq|y|\leq 3/(2|\xi|)}.$$

$$|III| \lesssim \int_B |K_\varepsilon(x + \xi/(2|\xi|^2))|\, dx$$

Note that $B \subseteq x : 1/(2|\xi|) \leq |x| - 1/(2|xi|) \leq |x + \xi|/(2|\xi|^2)| \leq 1/|\xi|$.

$$\lesssim \int_{1/(2|\xi|)\leq|y|\leq 1/|\xi|} |K_\varepsilon(y)|\, dy \lesssim 1.$$

So $\|\widehat{K}_\varepsilon\|_\infty \lesssim 1$, uniformly in $\varepsilon > 0$.

We claim that for $f \in \mathcal{S}(\mathbb{R})$, $\{K_\varepsilon * f\}_\varepsilon$ is Cauchy ni L^2. Assuming the claim, for $f \in \mathcal{S}(\mathbb{R}^d)$, let $K * f$ be the L^2 limit of $K_\varepsilon * f$. Then

$$\|K * f\|_2 \le \underbrace{\|K_\varepsilon * f\|_2}_{\lesssim \|f\|_2} + \underbrace{\|K * f - K_\varepsilon * f\|_2}_{\xrightarrow{\varepsilon \to 0} 0}.$$

So we have that

$$\|K * f\|_2 \lesssim \|f\|_2 + o(1)$$

as $\varepsilon \to 0$. Let $\varepsilon \to 0$ to get $\|K * f\|_2 \lesssim \|f\|_2$. For $f \in L^2$, let $f_n \in \mathcal{S}$ be such that $f_n \xrightarrow{L^2} f$. Then $\{f_n\}_n$ is Cauchy in L^2, so $\{K * f_n\}_{n \ge 1}$ is Cauchy in L^2. Let $K * f$ be the L^2-limit of $K * f_n$. Now

$$\|K * f\|_2 = \lim_n \|K * f\|_2 \lesssim \lim_n \|f_n\|_2 = \|f\|_2.$$

Now let's prove the claim: Fix $f \in \mathcal{S}(\mathbb{R}^d)$ and $0 < \varepsilon_1 < \varepsilon_2 < 1$. Then

$$\begin{aligned}
(K_{\varepsilon_1} * f - K_{\varepsilon_2} * f)(x) &= \int_{\varepsilon_1 \le |y| \le 1/\varepsilon_1} K(y) f(x-y)\, dy - \int_{\varepsilon_2 \le |y| \le 1/\varepsilon_2} K(y) f(x-y)\, dy \\
&= \int_{\varepsilon_1 \le |y| \le \varepsilon_2} K(y) f(x-y)\, dy + \int_{1/\varepsilon_2 \le |y| \le 1/\varepsilon_1} K(y) f(x-y)\, dy
\end{aligned}$$

Using property (b),

$$\begin{aligned}
\left| \int_{\varepsilon_1 \le |y| \le \varepsilon_2} K(y) f(x-y)\, dy \right| &= \left| \int_{\varepsilon_1 \le |y| \le \varepsilon_2} K(y) [f(x) - f(y)])\, dy \right| \\
&\le \int_{\varepsilon_1 \le |y| \le \varepsilon_2} |K(y)| |y| \int_0^1 |\nabla f(x - \theta y)|\, d\theta\, dy
\end{aligned}$$

Using property (a),

$$\begin{aligned}
&\lesssim \int_{\varepsilon_1 \le |y| \le \varepsilon_2} |y|^{1-d} \int \underbrace{|\nabla f(x - \theta y)|}_{\lesssim 1/\langle x - \theta y \rangle^d \lesssim 1/\langle x \rangle^d}\, d\theta\, dy \\
&\lesssim (\varepsilon_2 - \varepsilon_1) \frac{1}{\langle x \rangle^d}.
\end{aligned}$$

Alternatively, we could say

$$\begin{aligned}
\left\| \int_{\varepsilon_1 \le |y| \le \varepsilon_2} |y|^{1-d} \int |\nabla f(x - \theta y)|\, d\theta\, dy \right\|_{L^2_x} &\lesssim \int_{\varepsilon_1 \le |y| \le \varepsilon_2} |y|^{1-d} \int \|\nabla f(x - \theta y)\|_{L^2_x}\, d\theta\, dy \\
&\lesssim \|\nabla f\|_{L^2} (\varepsilon_2 - \varepsilon_1)
\end{aligned}$$

$$\xrightarrow{\varepsilon_2,\varepsilon_1\to 0} 0.$$

For the other term, using Young's inequality, we have

$$\begin{aligned}
\left\| \int_{1/\varepsilon_2\le|y|\le 1/\varepsilon_1} \int K(y)f(x-y)\,dy \right\|_{L^2_x} &\lesssim \|K\mathbb{1}_{\{1/\varepsilon_2\le|y|\le1/\varepsilon_1\}}\|_2\cdot\|f\|_1 \\
&\lesssim \|f\|_{L^1}\left(\int_{|y|\ge 1/\varepsilon_2} |y|^{-2d}\,dy\right)^{1/2} \\
&\lesssim \|f\|_1\varepsilon_2^{d/2} \\
&\xrightarrow{\varepsilon_2\to 0} 0. \qquad \square
\end{aligned}$$

Remark 16.1. The same argument show that for $f\in\mathcal{S}(\mathbb{R}^d)$, $\{K_\varepsilon * f\}_{\varepsilon>0}$ is Cauchy in L^p for $1<p<\infty$. It uses conditions (a), (b).

16.2 L^p bounds for Calderón-Zygmund convolution kernels

Theorem 16.2. *Let $K:\mathbb{R}^d\setminus\{0\}\to\mathbb{C}$ be a Calderón-Zygmund convolution kernel. For $\varepsilon>0$, let $K_\varepsilon = K\mathbb{1}_{\{\varepsilon\le|x|\le1/\varepsilon\}}$. Then*

1. *$|\{x:|K_\varepsilon * f|(x)>\lambda\}|\lesssim\frac{1}{\lambda}\|f\|_1$ uniformly in $\lambda>0, f\in L^1,\varepsilon>0$.*
2. *For any $1<p<\infty$, $\|K_\varepsilon * f\|_p\lesssim\|f\|_p$ uniformly for $f\in L^p,\varepsilon>0$.*

Consequently, $f\mapsto K*f$ (the L^p-limit of $K_\varepsilon * f$) extends continuous ly from $\mathcal{S}(\mathbb{R}^d)$ to a bounded map on L^p when $1<p<\infty$.

Proof. First, assume that we have proven the first claim. By the Marcinkiewicz interpolation theorem, we get the second claim for $1<p<2$. Now fix $2<p<\infty$. By duality,

$$\begin{aligned}
\|K_\varepsilon * f\|_p &= \sup_{\|g\|_{p'}=1}\langle K_\varepsilon * f, g\rangle \\
&= \sup_{\|g\|_{p'}=1}\langle f, \overline{K_\varepsilon^R} * g\rangle \\
&\lesssim \|f\|_p \sup_{\|g\|_{p'}=1}\|\overline{K_\varepsilon^R} * g\|_{p'} \\
&\lesssim \|f\|_p. \qquad \square
\end{aligned}$$

We will prove the first claim last time.

17 L^p Bounds for Calderón-Zygmund Convolution Kernels

17.1 Weak L^p bound for Calderón-Zygmund convolution kernels

Theorem 17.1. *Let $K : \mathbb{R}^d \setminus \{0\} \to \mathbb{C}$ be a Calderón-Zygmund convolution kernel. For $\varepsilon > 0$, let $K_\varepsilon = K \mathbb{1}_{\{\varepsilon \le |x| \le 1/\varepsilon\}}$. Then*

1. *$|\{x : |K_\varepsilon * f|(x) > \lambda\}| \lesssim \frac{1}{\lambda}\|f\|_1$ uniformly in $\lambda > 0, f \in L^1, \varepsilon > 0$.*
2. *For any $1 < p < \infty$, $\|K_\varepsilon * f\|_p \lesssim \|f\|_p$ uniformly for $f \in L^p, \varepsilon > 0$.*

Consequently, $f \mapsto K * f$ (the L^p-limit of $K_\varepsilon * f$) extends continuoussly from $\mathcal{S}(\mathbb{R}^d)$ to a bounded map on L^p when $1 < p < \infty$.

Proof. Assuming that (1) holds, we proved (2) using interpolation and duality. To show the last claim, it suffices to prove that $\{K_\varepsilon * f\}_{\varepsilon > 0}$ forms a Cauchy sequence in L^p $(1 < p < \infty)$ whenever $f \in \mathcal{S}(\mathbb{R}^d)$. We want to prove this using the L^2 result and condition (c) of the Calderón-Zygmund kernel; this will let our theory have more adaptability.

For $1 < p < 2$, let $1 < q < p$. Write $\frac{1}{p} = \frac{\theta}{q} + \frac{1-\theta}{2}$ for some $\theta \in (0,1)$. Then

$$\|K_{\varepsilon_1} * f - K_{\varepsilon_2} * f\|_p \lesssim \underbrace{\|K_{\varepsilon_1} * f + K_{\varepsilon_2} f\|_2^{1-\theta}}_{\xrightarrow{\varepsilon_1,\varepsilon_2 \to 0} 0} \underbrace{\|K_{\varepsilon_1} * f + K_{\varepsilon_2} f\|_q^{\theta}}_{\le (\|K_{\varepsilon_1} * f\|_q + \|K_{\varepsilon_2} * f\|_q)^\theta \lesssim \|f\|_q^\theta}$$
$$\xrightarrow{\varepsilon_1,\varepsilon_2 \to 0} 0.$$

For $2 < p < \infty$¡ let $p < r < \infty$ and write $\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{r}$. Then

$$\|K_{\varepsilon_1} * f - K_{\varepsilon_2} * f\|_p \le \underbrace{\|K_{\varepsilon_1} * f - K_{\varepsilon_2} * f\|_2^{1-\theta}}_{\xrightarrow{\varepsilon_1,\varepsilon_2 \to 0} 0} \underbrace{\|K_{\varepsilon_1} * f - K_{\varepsilon_2} * f\|_r^{\theta}}_{\lesssim \|f\|_r^\theta}$$

Let's show (1). For $\lambda > 0$, $f \in L^1$, and $\varepsilon > 0$, perform a Calderón-Zygmund decomposition for f at level λ: $f = g + b$ with $\operatorname{supp} b = \bigcup Q_k$, Q_k^o pairwise disjoint, and $\sum_k |Q_k| \le \|f\|_1/\lambda$. We can take

$$g(x) = \begin{cases} f(x) & x \notin \bigcup Q_k \\ \frac{1}{|Q_k|}\int_{Q_k} f(y)\,dy & x \in Q_k^o. \end{cases}$$

Then $|g| \lesssim \lambda$, and $b(x) = f(x) - \frac{1}{|Q_k|}\int_{Q_k} f(y)\,dy$ for $x \in Q_k$, so

$$\int_{Q_k} b(x)\,dx = 0, \qquad \frac{1}{|Q_k|}\int_{Q_k} |b(y)| \lesssim \lambda.$$

Then

$$|\{x : |K_\varepsilon * f|(x) > \lambda\}| \le |\{x : |K_\varepsilon * g|(x) > \lambda/2\}| + |\{x : |K_\varepsilon * b|(x) > \lambda/2\}|$$

$$\lesssim \frac{1}{\lambda^2}\|K_\varepsilon * g\|_2^2 + \left|\bigcup_k \alpha Q_k\right| + \left|\{x \in \left[\bigcup \alpha Q_k\right]^c : |K_\varepsilon * b|(x) > \lambda/2\}\right|$$

We have

$$\frac{1}{\lambda^2}\|K_\varepsilon * g\|_2^2 \lesssim \frac{\|g\|_2^2}{\lambda^2} \lesssim \frac{\lambda\|g\|_1}{\lambda^2} \lesssim \frac{\|f\|_1}{\lambda}$$

and

$$\left|\bigcup \alpha Q_k\right| \le \sum |\alpha Q_k| \le \alpha^d \sum |Q_k| \lesssim \alpha^d \frac{\|f\|_1}{\lambda}.$$

We are left with $E := |\{x \in [\bigcup \alpha Q_k]^c : |K_\varepsilon * b|(x) > \lambda/2\}|$. Let $x \notin \bigcup \alpha Q_k$. Then

$$\begin{aligned} K_\varepsilon * b(x) &= \int K_\varepsilon(x-y)b(y)\,dy \\ &= \sum_k \int_{Q_k} K_\varepsilon(x-y)b(y)\,dy \end{aligned}$$

Here, we only have a convolution, not an average. But a convolution is only as smooth as its smoothest term. So we have to use the regularity of K_ε (condition (c)).

$$= \sum_k \int_{Q_k} [K_\varepsilon(x-y) - K_\varepsilon(x-x_k)]b(y)\,dy.$$

Using Chebyshev,

$$\begin{aligned} E &\lesssim \frac{1}{\lambda}\int_{x \notin \bigcup \alpha Q_k} (K_\varepsilon * b)(x) \\ &\lesssim \frac{1}{\lambda}\sum_k \int_{x \in (\alpha Q_k)^c} \int_{Q_k} |K_\varepsilon(x-y) - K_\varepsilon(x-x_k)||b(y)|\,dy\,dx \end{aligned}$$

Change variables.

$$\lesssim \frac{1}{\lambda}\sum_k \int_{Q_k} |b(y)| \left(\int_{(\alpha Q_k)^c - \{x_k\}} |K_\varepsilon(x+x_k-y) - K_\varepsilon(x)|\,dx\right)\,dy.$$

For $y \in Q_k$, $|x_k - y| \le \frac{1}{2}\ell(Q_k)\sqrt{d}$. So we need $\alpha\ell(Q_k)/2 \ge 2\frac{1}{2}\ell(Q_k)\sqrt{d}$. So take $\alpha \ge 2\sqrt{d}$. Then using the regularity condition (c) of the convolution kernel, we get

$$\begin{aligned} E &\lesssim \frac{1}{\lambda}\sum_k \int_{Q_k} |b(y)| \cdot 1\,dy \\ &\lesssim \frac{\|f\|_1}{\lambda}. \end{aligned}$$

□

Remark 17.1. Once we have boundedness in L^2, the only condition we need to deduce boundedness in L^p for $1 < p < \infty$ is the regularity condition (c).

17.2 Application: The Hilbert transform

Here is an application.

Example 17.1. Let $K : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be $K(x) = \frac{1}{\pi x}$. This is a Calderón-Zygmund convolution kernel. So the **Hilbert transform**,

$$Hf(x) = \frac{1}{\pi} \int \frac{f(x-y)}{y} \, dy = \lim_{\varepsilon \to 0} \int_{|y|>\varepsilon} \frac{f(x-y)}{y} \, dy.,$$

is bounded on L^p for $1 < p < \infty$.

Remark 17.2. Boundedness on L^1 and L^∞ may fail. Consider the Hilbert transofrm, and take $f = \mathbb{1}_{[a,b]} \in L^1 \cap L^\infty$; we will show that $Hf \notin L^1 \cup L^\infty$. For $\varepsilon > 0$,

$$\begin{aligned} H_\varepsilon f(x) &:= \frac{1}{\pi} \int_{\varepsilon \le |y| \le 1/\varepsilon} \frac{\mathbb{1}_{[a,b]}(x-y)}{y} \, dy \\ &= \frac{1}{\pi} \int_{\substack{\varepsilon \le |y| \le 1/\varepsilon \\ x-b \le y \le x-a}} \frac{1}{y} \, dy \\ &= \frac{1}{\pi} \log \left| \frac{x-a}{x-b} \right| \end{aligned}$$

almost everywhere. But $Hf \notin L^1 \cup L^\infty$.

Remark 17.3. We have $\widehat{Hf}(\xi) = -i \operatorname{sgn}(\xi) \cdot \widehat{f}(\xi)$. For $a > 0$, let

$$\widehat{f}_a(\xi) = \begin{cases} e^{-a\xi} & \xi \ge 0 \\ 0 & \xi < 0, \end{cases} \qquad \widehat{g}_a(\xi) = \begin{cases} 0 & \xi > 0 \\ e^{q\xi} & \xi \le 0. \end{cases}$$

Then

$$(\widehat{f}_a - \widehat{g}_a)(\xi) = \begin{cases} e^{-a\xi} & \xi > 0 \\ 0 & \xi = 0 \\ -e^{a\xi} & \xi < 0 \end{cases} \xrightarrow{\mathcal{S}'(\mathbb{R}),\, a \to 0} \begin{cases} 1 & \xi > 0 \\ 0 & \xi = 0 \\ -1 & \xi < 0 \end{cases} = \operatorname{sgn}(\xi).$$

So we get

$$f_a - g_a \xrightarrow{\mathcal{S}'(\mathbb{R}),\, a \to 0} \operatorname{sgn}^\vee.$$

Next time, we will complete this computation.

18 The Mikhlin Multiplier Theorem

18.1 The Hilbert transform

Recall the **Hilbert transform**

$$Hf(x) = PV \int \frac{f(x-y)}{\pi y}\, dy.$$

We claimed that $\widehat{Hf}(\xi) = -i\,\mathrm{sgn}(\xi)\widehat{f}(\xi)$. Let

$$\widehat{f}_a(\xi) = \begin{cases} e^{-a\xi} & \xi > 0 \\ 0 & \xi \le 0, \end{cases} \qquad \widehat{g}_a(\xi) = \begin{cases} 0 & \xi > 0 \\ e^{a\xi} & \xi \le 0 \end{cases}$$

Then

$$(\widehat{f}_a - \widehat{g}_a)(\xi) = \begin{cases} e^{-a\xi} & \xi > 0 \\ 0 & \xi = 0 \\ -e^{a\xi} & \xi < 0 \end{cases} \xrightarrow{\mathcal{S}'(\mathbb{R}),\, a\to 0} \begin{cases} 1 & \xi > 0 \\ 0 & \xi = 0 \\ -1 & \xi < 0 \end{cases} = \mathrm{sgn}(\xi).$$

So we get

$$f_a(x) - g_a(x) \xrightarrow{\mathcal{S}'(\mathbb{R}),\, a\to 0} \mathrm{sgn}^{\vee}.$$

Now compute

$$f_a(x) = \int_0^\infty e^{2\pi i x\xi} e^{-a\xi}\, d\xi = \frac{1}{a - 2\pi i x},$$

$$g_a(x) = \int_{-\infty}^0 e^{2\pi i x\xi} e^{a\xi}\, d\xi = \frac{1}{a + 2\pi i x},$$

so

$$f_a(x) - g_a(x) = \frac{4\pi i x}{a^2 + 4\pi^2 x^2}.$$

Let $\varphi \in \mathcal{S}(\mathbb{R}^d)$, and compute

$$\lim_{a\to 0}(f_a - g_a)(\varphi) = \lim_{a\to 0} \int \frac{4\pi i x}{a^2 + 4\pi^2 x^2}\varphi(x)\, dx$$

We can't pull in the limit as is. We need the integrand to vanish near 0.

$$\begin{aligned} &= \lim_{a\to 0} \int \frac{4\pi i x}{a^2 + 4\pi^2 x^2}[\varphi(x) - \varphi(0)\mathbb{1}_{[-\varepsilon,\varepsilon]}(x)]\, dx \\ &= \lim_{a\to 0} \int_{-\varepsilon}^{\varepsilon} \frac{4\pi i x}{a^2 + 4\pi^2 x^2}[\varphi(x) - \varphi(0)] + \lim_{a\to 0} \int_{|x|>\varepsilon} \frac{4\pi i x}{a^2 + 4\pi^2 x^2}\varphi(x)\, dx \end{aligned}$$

$$= \int_{-\varepsilon}^{\varepsilon} \frac{i}{\pi x}[\varphi(x) - \varphi(0)] + \underbrace{\int_{|x|>\varepsilon} \frac{i}{\pi x}\varphi(x)\,dx}_{\xrightarrow{\varepsilon\to 0} i\,\mathrm{PV}(\frac{1}{\pi x})(\varphi)}.$$

On the other hand,

$$\left|\int_{-\varepsilon}^{\varepsilon} \frac{i}{\pi x}(\varphi(x) - \varphi(0))\,dx\right| \lesssim \varepsilon \|\varphi\|_{L^\infty} \xrightarrow{\varepsilon\to 0} 0.$$

So

$$f_a - g_a \xrightarrow{\mathcal{S}'(\mathbb{R}), a\to 0} i\,\mathrm{PV}\left(\frac{1}{\pi x}\right)(\varphi) = iH,$$

where $i\widehat{H} = \mathrm{sgn}$.

18.2 Littlewood-Paley projections and the Mikhlin multiplier theorem

Let's construct a dyadic partition of unity. Let $\varphi : \mathbb{R}^d \to [0,1]$, $\varphi \in C_c^\infty$ with

$$\varphi(x) = \begin{cases} 1 & |x| \le 1.4 \\ 0 & |x| > 1.42. \end{cases}$$

Let $\psi(x) = \varphi(x) - \varphi(2x)$; if we graph ψ, it is 0 before 0.7, increases quickly to 1 between 0.7 and 0.71, plateaus on 0.71 to 1.4, and goes down to 0 by 1.42.

For $N \in 2^{\mathbb{Z}}$, let $\psi_N 9x) = \psi(x/N)$. Note that

$$\sum_{N\in 2^{\mathbb{Z}}} \psi_N(x) = 1$$

a.e. (in fact for all $x \neq 0$.

Definition 18.1. The **Littlewood-Paley projection** to frequencies $|\xi| \sim N$ is given by

$$\widehat{P_N f}(\xi) = \widehat{f}(\xi)\psi_N(\xi), \qquad \text{i.e. } P_N f = f * [N^d \psi^\vee(N\,\cdot)].$$

We also define

$$\widehat{P_{\le N} f}(\xi) = \widehat{f}(\xi)\varphi(\xi/n), \qquad \text{i.e. } P_{\le N} f = [N^d \varphi^\vee(N\,\cdot)] * f$$

Remark 18.1. Caution: P_N is not a true projection since $P_N^2 = P_N$.

We can also define

$$P_{>n} = \mathrm{Id} - P_{\le N}, \qquad P_{M\le \cdot \le N} = \sum_{M\le K\le N} P_K.$$

Theorem 18.1 (Mikhlin multiplier theorem)**.** *Let $m : \mathbb{R}^d \backslash \{0\} \to \mathbb{C}$ be such that $|D_\xi^\alpha m(\xi)| \lesssim |\xi|^{-|\alpha|}$ uniformly for $|\xi| \neq 0$ and $0 \le |\alpha| \le \lceil \frac{d+1}{2} \rceil$. Then*

$$f \mapsto [m(\xi)\widehat{f}(\xi)]^\vee = m^\vee * f$$

is bounded on L^p for all $1 < p < \infty$.

Proof. Taking $\alpha = 0$, we get $M \in L^\infty$. By Plancherel,

$$\|m^\vee * f\|_2 = \|m\widehat{f}\|_2 \le \|m\|_{L^\infty} \|\widehat{f}\|_2 \lesssim \|f\|_2.$$

It suffices to check the regularity condition (c) is satisfied by the kernel $m^\vee$. We'll first prove this assuming $|D_\xi^\alpha m(\xi)| \lesssim |\xi|^{-|\alpha|}$ for $0 \le |\alpha| \le d+2$. In this case, we will show that $|\nabla m^\vee(x)| \lesssim |x|^{-(d+1)}$ uniformly for $|x| \neq 0$. This yields (c).

We have

$$\| |x^\alpha| \nabla m^\vee(x) \|_{L_x^\infty} \lesssim \underbrace{\left\| D_\xi^\alpha [\xi m(\xi)] \right\|_{L_\xi^1}}_{O(|\xi|^{1-|\alpha|})},$$

But this is not integrable! However, we can integrate it on dyadic annuli. Write

$$m(\xi) = \sum_{N \in 2^{\mathbb{Z}}} m_N(\xi), \qquad m_N(\xi) = m(\xi)\psi_N(\xi).$$

Then the chain rule gives

$$D_\xi^\alpha [\xi m_N(\xi)] = \sum_{\alpha_1 + \alpha_2 = \alpha} D_\xi^{\alpha_1} [\xi m(\xi)] D_\xi^{\alpha_2} [\psi_N(\xi)],$$

so

$$|D_\xi^\alpha [\xi m_N(\xi)]| \lesssim_\alpha \sum_{\alpha_1 + \alpha_2 = \alpha} |\xi|^{1-|\alpha_1|} N^{-|\alpha_2|} |D_\xi^{\alpha_2} \psi|(\xi/N).$$

Then

$$\begin{aligned} \| |x^\alpha| \nabla m_N^\vee(x) \|_{L_x^\infty} &\lesssim \| D_\xi^\alpha [\xi m_n(\xi)] \|_{L_\xi^1} \\ &\lesssim_\alpha \sum_{\alpha_1 + \alpha_2 = \alpha} \int_{|\xi| \sim N} |\xi|^{1-|\alpha_1|} N^{-|\alpha_2|} \, d\xi \\ &\lesssim_\alpha N^{1-|\alpha|+d}. \end{aligned}$$

So we get

$$|\nabla m_N^\vee(x)| \lesssim \min\{N^{d+1}, (N|x|^{d+2})^{-1}\}.$$

By the triangle inequality,

$$\begin{aligned}|\nabla m^\vee(x)| &\le \sum_{N\in 2^{\mathbb{Z}}} |\nabla m_N^\vee(x)| \\ &\lesssim \sum_{N\le |x|^{-1}} N^{d+1} + \sum_{N>|x|^{-1}} \frac{1}{N|x|^{d+2}} \\ &\lesssim |x|^{-(d+1)},\end{aligned}$$

uniform in $|x|\neq 0$.

Now let's prove condition (c) assuming this hypothesis holds for only $0\le|\alpha|\le\lceil\frac{d+1}{2}\rceil$. Look at

$$\int_{|x|\ge 2|y|} |m^\vee(x+y)-m^\vee(x)|\,dx = \sum_{N\in 2^{\mathbb{Z}}} \int_{|x|\ge 2|y|} |m_N^\vee(x+y)-m_N^\vee(x)|\,dx$$

If we have $\widehat{f}_N(\xi)=\widehat{f}(\xi)\psi_(\xi)$, then $f_N(x)=(f*N^d\psi(N\,\cdot))(x)=\int f(x-y)N^d\psi^\vee(Ny)\,dy$, so $|f_N(x)|\lesssim\int|f(x-y)|N^d\frac{1}{\langle Ny\rangle^m}\,dy$.

$$\begin{aligned}&\lesssim \sum_{N\le |y|^{-1}} \int_{|x|\ge 2|y|} |m_N^\vee(x+y)-m_N^\vee(x)|\,dx \\ &\qquad + 2\sum_{N>|y|^{-1}} \int_{|x|\ge|y|} |M_n^\vee(x)|\,dx\end{aligned}$$

Using the fundamental theorem of calculus,

$$\begin{aligned}&\lesssim \sum_{N\le |y|^{-1}} \int_{|x|\ge 2|y|} |y|\cdot\int_0^1 |\nabla m_N^\vee(x+\theta y)|\,d\theta\,dx \\ &\qquad + 2\sum_{N>|y|^{-1}} \int_{|x|\ge|y|} |m_N^\vee(x)|\,dx.\end{aligned}$$

We will complete the proof next time. □

19 The Mikhlin Multiplier Theorem and Properties of Littlewood-Paley Projections

19.1 The Mikhlin multiplier theorem

Theorem 19.1 (Mikhlin multiplier theorem)**.** *Let* $m : \mathbb{R}^d \backslash \{0\} \to \mathbb{C}$ *be such that* $|D_\xi^\alpha m(\xi)| \lesssim |\xi|^{-|\alpha|}$ *uniformly for* $|\xi| \neq 0$ *and* $0 \leq |\alpha| \leq \lceil \frac{d+1}{2} \rceil$*. Then*

$$f \mapsto [m(\xi)\widehat{f}(\xi)]^\vee = m^\vee * f$$

is bounded on L^p *for all* $1 < p < \infty$*.*

Proof. By Plancherel and $m \in L^\infty$, we get boundedness on L^2. So it suffices to check regularity condition (c):

$$\int_{|x| \geq 2|y|} |m^\vee(x+y) - m^\vee(x)|\, dx \lesssim 1$$

uniformly in y. We have

$$\int_{|x| \geq 2|y|} |m^\vee(x+y) - m^\vee(x)|\, dx \lesssim \sum_{N \in 2^{\mathbb{Z}}} \int_{|x| \geq 2|y|} |m_N^\vee(x+y) - m_N^\vee(x)|\, dx$$

where $M_N = m\psi_N = m\psi(\,\cdot\,/N)$.

$$\begin{aligned} &\leq \sum_{N \leq |y|^{-1}} \int_{|x|>2|y|} |m_N^\vee(x-y) - m_N^\vee(x)|\, dx \\ &\quad + 2 \sum_{N > |y|^{-1}} \int_{|x| \geq |y|} |m_N^\vee(x)|\, dx \\ &\leq \sum_{N \leq |y|^{-1}} \int_{|x| \geq 2|y|} |y| \int_0^1 |\nabla m_N^\vee(x+\theta y)|\, d\theta\, dx \\ &\quad + 2 \sum_{N > |y|^{-1}} \int_{|x| \geq |y|} |m_N^\vee(x)|\, dx. \end{aligned}$$

Last time, we had pointwise bound on derivatives by assuming more conditions for more values of α. Here, instead, we will use Plancherel. By Plancherel,

$$\begin{aligned} \|(2\pi i x)^\alpha m_N^\vee(x)\|_{L_x^2} &= \|D_\xi^\alpha m_N\|_{L_\xi^2} \\ &= \sum_{\alpha_1+\alpha_2=\alpha} c_{\alpha_1,\alpha_2} \|D_\xi^{\alpha_1} m(\xi) \cdot \frac{1}{N^{|\alpha_2|}} (D_\xi^{\alpha_2}(\xi/N)\|_2 \end{aligned}$$

$$\lesssim_\alpha \sum_{\alpha_1+\alpha_2} \left(\int |\xi|^{-2|\alpha_1|} N^{-2|\alpha_2|} \, d\xi \right)^{1/2}$$
$$\lesssim N^{d/2-|\alpha|}$$

for all $0 \leq \alpha \leq \lceil \frac{d+1}{2} \rceil$. By Cauchy-Schwarz,

$$\int_{|x|\leq A} |m_N^\vee(x)| \, dx \leq \|m_N^\vee\|_2 A^{d/2} \lesssim (AN)^{d/2}.$$

Similarly,

$$\int_{|x|>A} |m_N^\vee(x)| \, dx \leq \|x^\alpha m_N^\vee\|_2 \left(\int_{|x|>A} |x|^{-2|\alpha|} \, dx \right)^{1/2}$$
$$\lesssim N^{d/2-|\alpha|} A^{d/2-|\alpha|},$$

provided $|\alpha| > d/2$. So for $\alpha = \lceil \frac{d+1}{2} \rceil > d/2$, we get

$$\int_{|x|>A} |m_N^\vee(x)| \, dx \lesssim (NA)^{d/2-\lceil (d+1)/2 \rceil}.$$

Then

$$\sum_{|x|>|y|^{-1}} \int_{|x|\geq |y|} |m_N^\vee(x)| \, dx \lesssim \sum_{N>|y|^{-1}} (N|y|)^{d/2-\lceil (d+1)/2 \rceil}$$

This is a geometric series, so it is smaller than a constant times its largest term.

$$\lesssim 1,$$

uniformly in $y \in \mathbb{R}^d$. Taking $A = N^{-1}$ in our relations, we get

$$\int |m_N^\vee(x)| \lesssim 1,$$

uniformly in N.

The same arguments would give

$$\int |\nabla m_N^\vee(x)| \, dx \lesssim N,$$

uniformly in N. Indeed,

$$\|(2\pi i x)^\alpha \nabla m_N^\vee\|_2 = \|D^\alpha (\xi m_N)\|_2$$

$$
\begin{aligned}
&\lesssim_\alpha \sum_{\alpha_1+\alpha_2=\alpha} \left(\int_{|\xi|\sim N} |\xi|^{2-2|\alpha_1|} N^{-2|\alpha_2|} \, d\xi \right)^{1/2} \\
&\lesssim N^{1+d/2-|\alpha|},
\end{aligned}
$$

so we get

$$
\begin{aligned}
\int_{|x|\le A} |\nabla| m_N^\vee| &\lesssim N^{1+d/2} A^{d/2} = n(NA)^{d/2}, \\
\int_{|x|\ge A} |\nabla m_N^\vee| &\lesssim N(NA)^{d/2-\lceil (d+1)/2 \rceil}.
\end{aligned}
$$

We can now estimate

$$
\sum_{N\le |y|^{-1}} |y| \int_{|x|\ge 2|y|} \int_0^1 |\nabla m_N^\vee(x+\theta y)| \, d\theta \, dx \lesssim \sum_{N \le |y|^{-1}} |y| \cdot N \lesssim 1,
$$

uniformly in y. □

19.2 Properties of Littlewood-Paley projections

Recall the Littlewood-Paley projections:

$$
\varphi(x) = \begin{cases} 1 & |x| \le 1.4 \\ 0 & |x| > 1.42, \end{cases} \qquad \psi(x) = \varphi(x) - \varphi(2x).
$$

Then we had

$$
\begin{aligned}
f_N = P_N f &= f * N^d \psi^\vee(N\,\cdot), \\
f_{\le N} = P_{\le N} f &= f * N^d \varphi^\vee(N\,\cdot).
\end{aligned}
$$

Here are the basic properties of Littlewood-Paley projections.

Theorem 19.2.

1. *$\|f_n\|_p + \|f_{\le N}\|_p \lesssim \|f\|_p$ uniformly in N and for $1 \le p \le \infty$.*
2. *$|f_N(x)| + |f_{\le N}(x)| \lesssim Mf(x)$.*
3. *For $f \in L^p$ with $1 < p < \infty$, we have $f \overset{L^p}{=} \sum_{N\in 2^{\mathbb{Z}}} f_N$.*
4. *(Bernstein's inequality) For $1 \le p \le q \le \infty$,*

$$
\|f_N\|_q \lesssim N^{d/p-d/q} \|f_N\|_p
$$

$$
\|f_{\le N}\|_q \lesssim N^{d/p-d/q} \|f_{\le N}\|_p.
$$

5. *(Bernstein) For $1 \le p \le \infty$ and $s \in \mathbb{R}$,*

$$\||\nabla|^s f_N\|_p \sim N^s \|f_N\|_p.$$

In particular, for $s > 0$ and $1 \le p \le \infty$,

$$\||\nabla|^s f_{\le N}\|_p \lesssim N^s \|f_{\le N}\|_p.$$

$$\|\nabla|^{-s} f_{>N}\|_p \lesssim N^{-s} \|f_{>N}\|_p.$$

Proof.

1. By Young's inequality,

$$\begin{aligned}
\|f_N\|_p &= \|f * N^d \psi^\vee(N\,\cdot)\|_p \\
&\lesssim \|f\|_p \underbrace{\|N^d \psi^\vee(N\,\cdot)\|_1}_{} = \|\psi^\vee\|_1 \\
&\lesssim \|f\|_p,
\end{aligned}$$

$$\begin{aligned}
\|f_{\le N}\|_p &= \|f * N^d \varphi^\vee(N\,\cdot)\|_p \\
&\lesssim \|f\|_p \|\varphi_1^\vee\|_1 \\
&\lesssim \|f\|_p.
\end{aligned}$$

2.

$$\begin{aligned}
|f_N(x)| &\le \int |f(y) N^d \psi^\vee(N(x-y))|\,dy \\
&\lesssim N^d \int |f(y)| \frac{1}{\langle N(x-y)\rangle^{2d}}\,dy \\
&\lesssim N^d \int_{|x-y|\le 1/N} |f(y)|\,dy + \sum_{R\in 2^{\mathbb{Z}}} N^d \int_{R/N \le |x-y| \le 2R/N} \frac{|f(y)|}{R^{2d}}\,dy
\end{aligned}$$

How do we make this look like the maximal function?

$$\begin{aligned}
&\lesssim \frac{1}{|B(x,1/N)|} \int_{B(x,1/N)} |f(y)|\,dy \\
&\qquad + \sum_{R\in 2^{\mathbb{Z}}} R^{-d} \frac{1}{|B(x,2R/N)|} \int_{B(x,2R/N)} |f(y)|\,dy \\
&\lesssim Mf(x) \left[1 + \sum_{R\in 2^{\mathbb{Z}}} R^{-d}\right] \\
&\lesssim Mf(x).
\end{aligned}$$

3. First assume $f \in \mathcal{S}(\mathbb{R}^d)$. By Plancherel and dominated convergence,

$$\|f - P_{N \leq \cdot \leq 1/N} f\|_2 \xrightarrow{N \to 0} 0.$$

For $1 < p < 2$, write $\frac{1}{p} = \theta + \frac{1-\theta}{2} = \frac{1+\theta}{2}$.

$$\begin{aligned}\|f - P_{N \leq \cdot \leq 1/N} f\|_p &\leq \|f - P_{N \leq \cdot \leq 1/N} f\|_1^{\theta} \|f - P_{N \leq \cdot \leq 1/N} f\|_2^{1-\theta} \\ &\leq (\|f\|_1 + \|P_{N \leq \cdot \leq 1/N} f\|_1)^{\theta} \cdot \|f - P_{N \leq \cdot \leq 1/N} f\|_2^{1-\theta} \\ &\xrightarrow{N \to 0} 0\end{aligned}$$

by property (1). For $2 < p < \infty$,

$$\|f - P_{N \leq \cdot \leq 1/N} f\|_p \leq \underbrace{\| \quad \|_2^{2/p}}_{\xrightarrow{N \to 0} 0} \underbrace{\| \quad \|_\infty^{1-2/p}}_{\lesssim \|f\|_\infty}$$

If $f \in L^p$, let $g \in \mathcal{S}(\mathbb{R}^d)$ such that $\|f - g\|_p \leq \delta$. Then

$$\begin{aligned}\|f - P_{N \leq \cdot \leq 1/N} f\|_p &\lesssim \|g - P_{N \leq \cdot \leq 1/N} g\|_p + \|f - g\||_p + \|P_{N \leq \cdot \leq 1/N}(f - g)\|_p \\ &\lesssim o(1) + \delta\end{aligned}$$

as $N \to 0$. $\square$

We will prove (4) and (5) next time.

Remark 19.1. (3) fails for $p = 1$ and $p = \infty$. For $p = 1$, $\int P_N f = \widehat{P_n f}(0) = 0$, so pick some function with mean 0.

20 Littlewood-Paley Projections and Khinchine's Inequality

20.1 Bernstein properties of Littlewood-Paley projections

Last time, we were proving properties of Littlewood-Paley projections.

Theorem 20.1.

1. *$\|f_n\|_p + \|f_{\leq N}\|_p \lesssim \|f\|_p$ uniformly in N and for $1 \leq p \leq \infty$.*
2. *$|f_N(x)| + |f_{\leq N}(x)| \lesssim Mf(x)$.*
3. *For $f \in L^p$ with $1 < p < \infty$, we have $f \overset{L^p}{=} \sum_{N \in 2^{\mathbb{Z}}} f_N$.*
4. *(Bernstein's inequality) For $1 \leq p \leq q \leq \infty$,*
$$\|f_N\|_q \lesssim N^{d/p-d/q}\|f_N\|_p$$
$$\|f_{\leq N}\|_q \lesssim N^{d/p-d/q}\|f_{\leq N}\|_p.$$
5. *(Bernstein) For $1 \leq p \leq \infty$ and $s \in \mathbb{R}$,*
$$\||\nabla|^s f_N\|_p \sim N^s\|f_N\|_p.$$
In particular, for $s > 0$ and $1 \leq p \leq \infty$,
$$\||\nabla|^s f_{\leq N}\|_p \lesssim N^s\|f_{\leq N}\|_p.$$
$$\|f_{>N}\|_p \lesssim N^{-s}\||\nabla|^s f_{>N}\|_p.$$

We proved properties (1) to (3) last time.

Proof. Here is 4: We have $f_N = f * N^d\psi^\vee(N\,\cdot)$, so by Young's inequality,

$$\begin{aligned}\|f_n\|_q &\lesssim \|f\|_p \cdot \|N^d\psi^\vee(N\,\cdot)\|_{qp/(qp+p-q)} \\ &\lesssim \|f\|N^{d-d(1+1/q-1/p)} \\ &\lesssim N^{d/p-d/q}\|f\|_p\end{aligned}$$

To recover f_N on the RHS, we use a common trick. Let $\widetilde{\psi}(\xi) = \psi(2\xi) + \psi(\xi) + \psi(\xi/2)$, $\widetilde{\psi}_N(\xi) = \widetilde{\psi}(\xi/N)$, and define the fattened LIttlewood-Paley projection

$$\widehat{\widetilde{P}_N f}(\xi) = \widehat{f}(\xi) \cdot \widetilde{\psi}_N(\xi).$$

Note that $\widetilde{P}_N P_n = P_n$ since $\widetilde{\psi} \equiv 1$ on $\operatorname{supp}\psi$. Write

$$f_N = \widetilde{P}_N f = f_n * [N^d\widetilde{\psi}^\vee(N\,\cdot)]$$

and argue as before. The same argument gives $\|f_{\le N}\|_q \le N^{d/p-d/q}\|f_{\le N}\|_p$. (We use $P_{\le 4N}P_{\le N} = P_{\le N}$.)

Here is 5: Note that

$$\begin{aligned}
|\nabla|^s f_N &= [(2\pi|\xi|)^s \psi_N(\xi)]^\vee * f \\
&= N^s \left[\left(\frac{2\pi|\xi|}{N}\right)^s \psi(\xi/N))\right]^\vee * f.
\end{aligned}$$

Let

$$\chi(\xi) = (2\pi|\xi|)^s \psi(\xi) \in C_c^\infty(\mathbb{R}^d \setminus \{0\}), \qquad \chi_N(\xi) = \chi(\xi/N).$$

Then $|\nabla|^s f_N = N^s[N^d \chi^\vee(N\,\cdot)] * f$. So

$$\begin{aligned}
\||\nabla|^s f_N\|_p &\lesssim N^s \|f\|_p \underbrace{\|N^d \chi_\vee(N\,\cdot)\|}_{=\|\chi^\vee\|_1} \\
&\lesssim N^s \|f\|_p.
\end{aligned}$$

Using the fattened Littlewood-Paley projection P_N, we get

$$\||\nabla|^s f_N\|_p \lesssim N^s \|f_n\|_p.$$

On the other hand,

$$\|f_n\|_p = \||\nabla|^{-s}|\nabla|^s f_n\|_p \lesssim N^{-s}\||\nabla|^s f_N\|_p.$$

Finally, for $s > 0$,

$$\begin{aligned}
\||\nabla|^s f_{\le N}\|_p &\lesssim \sum_{M \le N} \||\nabla|^s f_M\|_p \\
&\lesssim \sum_{M \le N} M^s \underbrace{\|f_M\|_p}_{\lesssim \|f\|_p} \\
&\lesssim N^s \|f\|_p.
\end{aligned}$$

For high frequencies,

$$\begin{aligned}
\|f_{>N}\|_p &\lesssim \sum_{M > N} \|f_M\|_p \\
&\lesssim \sum_{M > N} M^{-s} \underbrace{\||\nabla|^s f_M\|_p}_{\lesssim \||\nabla|^s f\|_p} \\
&\lesssim N^{-s}\||\nabla|^s f\|_p.
\end{aligned}$$

□

20.2 Khinchine's inequality

Lemma 20.1 (Khinchine's inequality)**.** *Let $\{X_n\}_{n\geq 1}$ be independent, identically distributed random variables with $X_n = \pm 1$ with equal probability. Let $\{c_n\}_{n\geq 1} \subseteq \mathbb{C}$ and $0 < p < \infty$. Then*

$$\mathbb{E}\left[\left|\sum_{n\geq 1} c_n X_n\right|^p\right]^{1/p} \sim_p \sqrt{\sum_{n\geq 1} |c_n|^2}.$$

One way to think about this is that a random variable's "size" is given by its variance. For $p = 2$,

$$\begin{aligned}\mathbb{E}\left[\left|\sum_{n\geq 1} c_n X_n\right|^2\right] &= \mathbb{E}[\Big(\sum c_n X_n\Big)\Big(\sum \overline{c}_m X_m\Big)] \\ &= \sum |c_n|^2\,\mathbb{E}[X_n^2] + \sum_{n\neq m} c_n\overline{c}_m \underbrace{\mathbb{E}[X_n X_m]}_{\to 0} \\ &= \sum |c_n|^2.\end{aligned}$$

So this basically says that this orthogonality persists, even in an L^p sense.

Proof. Without loss of generality, we may assume $c_n \in \mathbb{R}$.

$$\mathbb{E}\left[\left|\sum_{n\geq 1} c_n X_n\right|^p\right] = p\int_0^\infty \lambda^p \mathbb{P}\left(\left|\sum c_n X_n\right| > \lambda\right)\frac{d\lambda}{\lambda}$$

By Chebyshev,

$$\begin{aligned}\mathbb{P}\left(\sum c_n X_n > \lambda\right) &\leq e^{-\lambda t}\,\mathbb{E}[e^{t\sum c_n X_n}] \\ &= e^{-\lambda t}\,\mathbb{E}\left[\prod_n e^{tc_n X_n}\right] \\ &= e^{-\lambda t}\prod_n \mathbb{E}[c^{tc_n X_n}] \\ &= e^{-\lambda t}\prod_n \frac{e^{tc_n} + e^{-tc_n}}{2} \\ &= e^{-\lambda t}\prod_n \cosh(tc_n)\end{aligned}$$

Use that $\cosh x \leq e^{x^2/2}$.

$$= e^{-\lambda t}\prod_n e^{t^2 c_n^2/2}$$

$$= e^{-\lambda t + t^2/2(\sum c_n^2)}.$$

Choose t such that $\lambda t = t^2 \sum c_n^2$; so $t = \lambda / \sum c_n^2$. We get

$$\mathbb{P}\left(\sum c_n X_n > \lambda\right) \leq e^{-\lambda^2/(2\sum c_n^2)}.$$

The same argument gives

$$\mathbb{P}\left(\sum c_n X_n < -\lambda\right) \leq e^{-\lambda t}\, \mathbb{E}[e^{-t\sum c_n X_n}] \leq e^{-\lambda^2/(2\sum c_n^2)}.$$

So we have

$$\begin{aligned}\mathbb{E}\left[\left|\sum_{n\geq 1} c_n X_n\right|^p\right] &= p\int_0^\infty \lambda^p \mathbb{P}\left(\left|\sum c_n X_n\right| > \lambda\right)\frac{d\lambda}{\lambda} \\ &\lesssim_p \int_0^\infty \lambda^p e^{-\lambda^2/(2\sum c_n^2)}\frac{d\lambda}{\lambda}\end{aligned}$$

Make the change of variables $\beta = \lambda/\sqrt{\sum c_n^2}$.

$$\lesssim_p \left(\sum c_n^2\right)^{p/2} \underbrace{\int_0^\infty \beta^p e^{-\beta^2/2}\frac{d\beta}{\beta}}_{\lesssim_p 1}.$$

For the other inequality, for $1 < p < \infty$,

$$\begin{aligned}\sum |c_n|^2 &= \mathbb{E}\left[\left|\sum c_n X_n\right|^2\right] \\ &\lesssim \mathbb{E}\left[\left|\sum c_n X_n\right|^p\right]^{1/p} \underbrace{\mathbb{E}\left[\left|\sum c_n X_n\right|^{p'}\right]^{1/p'}}_{\lesssim\sqrt{\sum |c_n|^2}},\end{aligned}$$

which gives us

$$\sqrt{\sum c_n^2} \lesssim \mathbb{E}\left[\left|\sum c_n X_n\right|^p\right]^{1/p}.$$

For $0 < p \leq 1$, we use Cauchy-Schwarz instead:

$$\begin{aligned}\sum |c_n|^2 &= \mathbb{E}\left[\left|\sum c_n X_n\right|^2\right] \\ &= \mathbb{E}\left[\left|\sum c_n X_n\right|^{p/2}\left|\sum c_n X_n\right|^{2-p/2}\right]\end{aligned}$$

$$\lesssim \mathbb{E}\left[\left|\sum c_n X_n\right|^p\right]^{1/2} \underbrace{\mathbb{E}\left[\left|\sum c_n X_n\right|^{4-p}\right]^{1/2}}_{\lesssim (\sum |c_n|^2)^{1/2 \cdot 1/2 \cdot (4-p)}}.$$

So we get that

$$\left(\sum |c_n|^2\right)^{p/4} \lesssim \mathbb{E}\left[\left|\sum c_n X_n\right|^p\right]^{1/2}.$$

Now raise both sides to the power $2/p$. □

20.3 Littlewood-Paley square function estimate

Theorem 20.2 (Littlewood-Paley square function estimate)**.** *Let $f \in \mathcal{S}(\mathbb{R}^d)$ and define the square function*

$$S(f) = \sqrt{\sum |f_N|^2}.$$

Then

$$\|S(f)\|_p \sim_p \|f\|_p \qquad \forall 1 < p < \infty.$$

Proof. Let's prove $\|Sf\|_p \lesssim_p \|f\|_p$. Let $\{X_N\}_{n \in 2^{\mathbb{Z}}}$ be iid random variables with $X_n = \pm 1$ with equal probability. Let

$$m_X(\xi) = \sum_{N \in 2^{\mathbb{Z}}} X_N \psi_N(\xi).$$

Note that

$$m_X^\vee * f = \sum_{N \in 2^{\mathbb{Z}}} X_N f_N.$$

We claim that m_X is a Mikhlin multiplier uniformly in the choice of X_N.

$$|D_\xi^\alpha m_X(\xi)| \lesssim \sum_{N \in 2^{\mathbb{Z}}} N^{|\alpha|} |D_\xi^\alpha \psi|(\xi/N)$$

Since ψ has compact support on $\mathbb{R} \setminus \{0\}$, only finitely many N contribute to the sum.

$$\lesssim |\xi|^{-\alpha}.$$

□

We will finish the proof next time.

Remark 20.1. We could replace ψ by any $C_c^\infty(\mathbb{R}^d \setminus \{0\})$ and still get a Mikhlin multiplier.

21 Estimates on the Littlewood-Paley Square Function and the Fractional Product Rule

21.1 Estimates on the Littlewood-Paley square function

Theorem 21.1 (Littlewood-Paley square function estimate). *Let $f \in \mathcal{S}(\mathbb{R}^d)$ and define the square function*

$$S(f) = \sqrt{\sum |f_N|^2}.$$

Then

$$\|S(f)\|_p \sim_p \|f\|_p \qquad \forall 1 < p < \infty.$$

Proof. Let $\{X_N\}_{N\in 2^{\mathbb{Z}}}$ be iid random variables with $X_n = \pm 1$ with equal probability, and define the random variable $m_X(\xi) = \sum X_n \psi_N(\xi)$. Last time, we showed that m_X is a Mikhlin multiplier, uniformly in the choice of X_N. This holds even if we replace ψ be another $C_c^\infty(\mathbb{R}^d \setminus \{0\})$ function. Now

$$m_X^\vee * f = \sum_{N\in 2^{\mathbb{Z}}} X_N f_N.$$

By Kinchine's inequality,

$$\mathbb{E}[|m_X^\vee * f|^2]^{1/p} \sim \sqrt{\sum |f_N|^2} \sim_p S(f).$$

Now

$$\begin{aligned}
\|S(f)\|_p^p &\sim \int \mathbb{E}[|m_X^\vee * f|^p(x)]\,dx \\
&\sim \mathbb{E}\Bigg[\underbrace{\int |m_X^\vee * f|^p(x)\,dx}_{\|m_X^\vee * f\|_p^p}\Bigg] \\
&\lesssim \mathbb{E}[\|f\|_p^p] \\
&\lesssim \|f\|_p^p.
\end{aligned}$$

Again, note that this holds for any $C_c^\infty(\mathbb{R}^d \setminus \{0\})$ function in place of ψ.

To prove the reverse inequality, we argue by duality and use the generality under which we proved the first inequality. We say

$$\begin{aligned}
\|f\|_p &= \sup_{\|g\|_{p'}=1} \langle f, g\rangle \\
&= \sup_{\|g\|_{p'}=1} \left\langle \sum P_N f, g \right\rangle
\end{aligned}$$

Since $\widetilde{P}_N P_N = P_N$ and $\widetilde{P}_n$ is self-adjoint,

$$= \sup_{\|g\|_{p'}=1} \sum_{N \in 2^{\mathbb{Z}}} \langle P_N f, \widetilde{P}_N g \rangle$$
$$\leq \sup_{\|g\|_{p'}=1} \int \sqrt{\sum_N |P_N f|^2} \sqrt{\sum_N |\widetilde{P}_N g|^2} \, dx$$

Using Hölder,

$$\leq \|S(f)|_p \sup_{\|g\|_{p'}=1} \left\| \sqrt{\sum_N |\widetilde{P}_N g|^2} \right\|_{p'}.$$

Replacing ψ by $\widetilde{\psi}(\xi) = \psi(2\xi) + \psi(\xi) + \psi(\xi/2) \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$ in the previous argument, we get

$$\left\| \sqrt{\sum_N |\widetilde{P}_N g|^2} \right\|_{p'} \lesssim \|g\|_{p'} \lesssim 1.$$

□

Corollary 21.1. *Fix* $1 < p < \infty$. *Then*

1. *Whenever* $s > -d$ *and* $f \in \mathcal{S}(\mathbb{R}^d)$ *(or* $s \in \mathbb{R}$ *and* $\widehat{f} \in C_c^\infty(\mathbb{R}^d \setminus \{0\})$*),*

$$\||\nabla|^s f\|_p \sim_p \left\| \sqrt{\sum N^{2s} |f_N|^2} \right\|_p.$$

2. *For* $s > 0$ *and* $f \in \mathcal{S}(\mathbb{R}^d)$,

$$\||\nabla|^s f\|_p \sim_p \left\| \sqrt{\sum N^{2s} |f_{\geq N}|^2} \right\|_p.$$

Proof.

1. Let's show that $\left\| \sqrt{\sum N^{2s} |f_N|^2} \right\|_p \lesssim \||\nabla|^s f\|_p$. We have

$$\sum N^{2s} |f_N|^2 = \sum N^{2s} ||\nabla|^{-s} |\nabla|^s f_N|^2$$
$$= \sum |N^s |\nabla|^{-s} P_N(|\nabla|^s f)|^2$$

Replacing ψ by $\chi(\xi) = \frac{1}{(2\pi|\xi|)^s} \psi(\xi) \in C_c^\infty(\mathbb{R}^d \setminus \{0\}$ and ψ_N by $\chi_N(\xi) = \left(\frac{N}{2\pi|\xi|}\right)^s \psi_N(\xi)$, we get

$$\left\| \sqrt{\sum |N^s |\nabla|^{-s} P_N(|\nabla|^s f)|^2} \right\|_p \lesssim_p \||\nabla|^s f\|_p.$$

To prove the reverse inequality, we argue by duality:

$$\begin{aligned}\||\nabla|^s f\|_p &= \sup_{\|g\|_{p'}=1} \langle |\nabla|^s f, g\rangle \\ &= \sup_{\|g\|_{p'}=1} \sum_N \left\langle |\nabla|^s f_N, \widetilde{P}_N g\right\rangle \\ &= \sup_{\|g\|_{p'}=1} \sum_N \left\langle N^s f_N, N^{-s}|\nabla|^s \widetilde{P}_N g\right\rangle\end{aligned}$$

Recall that $\mathcal{F} = \{h \in \mathcal{S}(\mathbb{R}^d) : \widehat{h} \text{ vanishes in a nbhd of } 0\}$ is dense in $L^{p'}$. So we can always take g to be in this family. So

$$\begin{aligned}\||\nabla|^s f\|_p &\leq \sup_{\substack{g\in\mathcal{F}\\ \|g\|_{p'}=1}} \int \sqrt{\sum N^{2s}|f_N|^2}\sqrt{\sum N^{-2s}||\nabla|^s \widetilde{P}_n g|^2}\, dx \\ &\leq \left\|\sqrt{\sum N^{2s}|f_N|^2}\right\|_p \sup_{\|g\|_{p'}=1} \left\|\sqrt{\sum N^{-2s}||\nabla|^s \widetilde{P}_N g|^2}\right\|_p .\end{aligned}$$

Replacing ψ by

$$\chi(\xi) = (2\pi|\xi|)^s \widetilde{\psi}(\xi), \qquad \chi_n(\xi) = \left(\frac{2\pi|\xi|}{N}\right)^s \widetilde{\psi}_N(\xi),$$

we get

$$\left\|\sqrt{\sum N^{-2s}||\nabla|^s \widetilde{P}_N g|^2}\right\|_{p'} \lesssim \|g\|_{p'} \lesssim 1.$$

2. We claim that $\sum N^{2s}|f_{\geq N}|^2 \sim \sum N^{2s}|f_N|^2$. We have

$$\sum_N N^{2s}|f_{\geq N}|^2 = \sum_N N^{2s}\left(\sum_{N_1\geq N} f_{N_1}\right)\left(\overline{\sum_{N_2\geq N} f_{N_2}}\right)$$

By paying a factor of 2, we can assume $N_1 \leq N_2$.

$$\begin{aligned}&\leq 2 \sum_{N\leq N_1\leq N_2} N^{2s}|f_{N_1}|\cdot|f_{N_2}| \\ &\lesssim \sum_{N_1\leq N_2} N_1^{2s}|f_{N_1}||f_{N_2}| \\ &\lesssim \sum_{N_1\leq N_2} \left(\frac{N_1}{N_2}\right)^s (N_1^s|f_{N_1}|)\,(N_2^s|f_{N_2}|)\end{aligned}$$

By Cauchy-Schwarz,

$$\lesssim \sum_N N^{2s} |f_N|^2.$$

On the other hand,

$$|f_N| = |f_{\geq N} - f_{\geq 2N}| \leq |f_{\geq N} f_{\geq 2N}|.$$

So

$$\begin{aligned} \sum_N N^{2s} |f_N|^2 &\lesssim \sum_N N^{2s} |f_{\geq n}|^2 + 2^{-2s} \sum_N (2N)^{2s} |f_{\geq 2N}|^2 \\ &\lesssim \sum_N N^{2s} |f_{\geq N}|^2. \end{aligned}$$

□

21.2 The fractional product rule

Theorem 21.2 (Fractional product rule, Christ-Weinstein, 1991)**.** *Fix* $s > 0$ *and* $1 < p, p_1, p_2, q_1, q_2 < \infty,$. *Then*

$$\| |\nabla|^s (fg) \|_p \lesssim \| |\nabla|^s f \|_{p_1} \|g\|_{p_2} + \|f\|_{q_1} + \| |\nabla|^s g \|_{q_2}.$$

whenever $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{q_1} + \frac{1}{q_2}$.

Remark 21.1. p_2 and q_1 are allowed to be ∞.

We really should only be proving this for $0 < s < 1$, since for integers, we can just use the regular product rule and then look at the fractional part.

Proof. We have

$$\| |\nabla|^s (fg) \|_p \sim \left\| \sqrt{\sum_N N^{2s} |P_N(fg)|^2} \right\|_p.$$

We write $fg = f_{\geq N/4} g + f_{>N/4} g_{\geq N/4} + f_{<N/4} g_{<N/4}$, so

$$P_N(fg) = P_N(f_{\geq N/4} g) + P_N(f_{>N/4} g_{\geq N/4}) + \cancelto{0}{P_N(f_{<N/4} g_{<N/4})}.$$

This gives

$$\begin{aligned} |P_N(fg)| &\lesssim M(f_{\geq N/4} g) + M(f_{\leq N/4} g_{\geq N/4}) \\ &\lesssim M(f_{\geq N/4} g) + M((Mf) g_{\geq N/4}). \end{aligned}$$

So we get

$$\sum N^{2s}|P_N(fg)|^2 \lesssim \sum |M((N^s f_{\geq N/4})g)|^2 + \sum |((Mf)\cdot N^s g_{\geq N/4})|^2,$$

which gives

$$\sqrt{\sum N^{2s}|P_N(fg)|^2} \lesssim \sqrt{\sum |M((N^s f_{\geq N/4})g)|^2} + \sqrt{\sum |((Mf)\cdot N^s g_{\geq N/4})|^2}.$$

So we get

$$\||\nabla|^s(fg)\|_p \lesssim \left\|\sqrt{N^{2s}|f_{\geq}N/4|^2}g\right\|_p + \left\|Mf\sqrt{N^{2s}|g_{\geq}N/4|^2}\right\|_p$$

By the corollary,

$$\lesssim \||\nabla|^s f\|_{p_1}\|g\|_{p_2} + \|f\|_{q_1}\||\nabla|^s g\|_{q_2}. \qquad \square$$

22 The Fractional Chain Rule

22.1 Proof of the fractional chain rule

Theorem 22.1 (Fractional chain rule, Christ-Weinstein, 1991)**.** *Let $F : \mathbb{C} \to \mathbb{C}$ be such that*

$$|F(u) - F(v)| \le |u - v|[G(u) + G(v)],$$

where $G : \mathbb{C} \to [0, \infty)$. Then for $0 < s < 1$, $1 < p, p_1 < \infty$, $1 < p_2 \le \infty$ such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$,

$$\| |\nabla|^s (F \circ u) \|_p \lesssim \| |\nabla|^s u \|_{p_1} \cdot \| G \circ u \|_{p_2}.$$

Example 22.1. Consider some nonlinear interaction: Let $F(u) = |u|^p u$, where $p > 0$. Then

$$|F(u) - F(v)| \lesssim |u - v|[|u|^p + |v|^p],$$

so we get a bound.

Proof. Last time, we showed that

$$\| |\nabla|^s (F \circ u) \|_p \sim \left\| \sqrt{\sum N^{2s} |P_N (F \circ u)|^2} \right\|_p .$$

Let's calculate

$$[P_N (f \circ u)](x) = \int N^d \psi^\vee (Nu)(F \circ u)(x - y)\, dy$$

We want to isolate u in this expression. We will use the locally Lipschitz condition. Since $\int \psi^\vee \, dy = \psi(0) = 0$,

$$= \int N^d \psi_\vee (Ny)[(F \circ u)(x - y) - (F \circ u)(x)]\, dy.$$

So we have

$$|P_N (F \circ u)|(x) \le \int N^d |\psi^\vee (Ny)| \cdot |u(x - y) - u(x)|[(G \circ u)(x - y) + (G \circ u)(x)]\, dy$$

We expect cancellation in the u terms at low frequencies. So we decompose

$$|u(x - y) - u(x)| \le \underbrace{|u_{>N}(x - y)|}_{I} + \underbrace{|u_{>N}(x)|}_{II} + \underbrace{\sum_{k \le N} |u_k (x - y) - u_k (x)|}_{III} .$$

Let's consider the contribution of I:

$$\int N^d |\psi^\vee (Ny)| u_{>N}(x - y)|[(G \circ u)(x - y) + (G \circ u)(x)]\, dy$$

We can bound this using the maximal function. We have $\int N^d |\psi^\vee(Ny)||g(x-y)|\,dy \lesssim \int_{|y|\le 1/N} |g(x-y)|\,dy + \sum_{R\in 2^{\mathbb{N}}} \int_{R/N\le |y|\le 2R/N} N^d \frac{1}{R^{2d}} |g(x-y)|\,dy \lesssim \frac{1}{|B(0,1/N)|}\int_{B(0,1/N)} |g(x-y)|\,dy + \cdots$.

$$\begin{aligned}
&\lesssim M(u_{\geq N}(G\circ u))(x) + M(u_{>N})(x)(G\circ u)(x)\\
&\lesssim M(u_{>N}(G\circ u))(x) + M(u_{>N})(x)M(G\circ u)(x).
\end{aligned}$$

This contributes the following to the original estimate:

$$\begin{aligned}
&\left\|\sqrt{\sum N^{2s}|M(u_{>N}(G\circ u))|^2}\right\|_p + \left\|\sqrt{\sum N^{2s}|M(u_{>N})M(G\circ u)|^2}\right\|_p\\
&\lesssim \left\|\sqrt{\sum |M(N^s u_{>N})(G\circ u)|^2}\right\|_p + \left\|M(G\circ u)\sqrt{\sum |M(N^s u_{>N})|^2}\right\|_p
\end{aligned}$$

Using our bounds for the vector-valued maximal function and Hölder,

$$\begin{aligned}
&\lesssim \left\|\sqrt{\sum_N |N^s u_{>N}|^2}(G\circ u)\right\| + \left\|\sqrt{\sum |N^s u_{>N}|^2}\right\|_{p_1} \|M(G\circ u)\|_{p_2}\\
&\lesssim \||\nabla|^s u\|_{p_1}\cdot \|G\circ u\|_{p_2}.
\end{aligned}$$

This is an acceptable contribution for what we want to prove.

Let's look at what II. To $P_N(F\circ u)$, this contributes

$$\begin{aligned}
&\int N^d|\psi^\vee(Ny)|u_{>N}(x)|[(G\circ u)(x-y) + (G\circ u)(x)]\,dy\\
&\lesssim |u_{>N}(x)|M(G\circ u)(x) + |u_{>N}(x)|(G\circ u)(x)\\
&\lesssim M(u_{>n})(x)M(G\circ u)(x).
\end{aligned}$$

As before, the contribution of II to the right hand side of the original estimate is acceptable.

We turn to III. We claim that

$$|u_k(x-y) - u_k(x)| \lesssim k|y|\cdot[M(u_k)(x-y) + Mu_k(x)]$$

We split into cases:

1. $k|y|>1$: Then

$$\begin{aligned}
|u_k(x-y)-u_k(x)| &\le |\widetilde{P}_k u_k|(x-y) + |\widetilde{P}_k u_k|(x)\\
&\lesssim (Mu_k)(x-y) + (Mu_k)(x).
\end{aligned}$$

2. $k|y|\le 1$:

$$|u_k(x-y)-u_k(x)| = \left|\int k^d \widetilde{\psi}^\vee(kz)[u_k(x-y-z) - u_k(x-z)]\,dz\right|$$

$$= \left| \int k^d [\widetilde{\psi}^\vee(k(z-y)) - \widetilde{\psi}^\vee(kz)] u_k(x-z)\, dz \right|$$

Using the fundamental theorem of calculus,

$$= \int k^d k|y| \int_0^1 \underbrace{|\nabla \widetilde{\psi}^\vee|(kz - \theta k y)}_{\lesssim 1/\langle kz-\theta ky\rangle^{2d} \lesssim 1/\langle kz\rangle^{2d}} |u_k(x-z)|\, d\theta\, dz$$

$$\lesssim k|y| \cdot (Mu_k)(x),$$

proving the claim.

To $P_N(f \circ u)$, the term III contributes

$$\int N^d |\psi^\vee(Ny)| \sum_{K \le N} K|y| [(Mu_k)(x-y) - M(u_k)(x)] \cdot [(G\circ u)(x-y) + (G \circ u)(x)]\, dy$$

$$\lesssim \sum_{k \le N} \frac{k}{N} \int N^d \frac{N|y|}{\langle N|y|\rangle^{3d}} [(Mu_k)(x-y) + (Mu_k)(x)] \cdot [(G\circ u)(x-y) + (G\circ u)(x)]\, dy$$

$$\lesssim \sum_{k \le N} \frac{k}{N} \cdot [M((Mu_k)\cdot (G\circ u))(x) + M(Mu_k)(x) \cdot M(G\circ u)(x)].$$

The contribution of III to the right hand side of the original estimate is

$$\lesssim \left\| \sqrt{\sum_N N^{2s} \left| \sum_{k \le N} \frac{k}{N} M((Mu_k)(G\circ u)) \right|^2} \right\|_p + \left\| \sqrt{N^{2s} \left| \sum_{k\le N} \frac{k}{N} M(Mu_k) \cdot M(G \circ u) \right|^2} \right\|_p$$

Both cases have terms like

$$\sum_N N^{2s} \left| \sum_{k \le N} \frac{k}{N} c_k \right|^2 \le 2 \sum_{k \le L \le N} N^{2s} \frac{kL}{NN} |c_k||c_L|$$

$$\lesssim \sum_{k \le L} L^{2s} \frac{k}{L} |c_k||c_L|$$

$$\lesssim \sum_{k \le L} \left(\frac{k}{L}\right)^{1-s} k^s |c_k| L^s |c_L|$$

By Cauchy-Schwarz (or Schur's test),

$$\lesssim \sqrt{\sum_k k^{2s} |c_k|^2} \sqrt{\sum_L L^{2s} |c_L|^2}$$

$$\lesssim \sum_N N^{2s}|c_N|^2.$$

And we use our maximal function bounds to finish the proof. □

23 Introduction to Oscillatory Integrals

23.1 Decay of integrals with compactly supported integrand

Oscillatory integrals are of two types

1. First kind:
$$I(\lambda) = \int e^{i\lambda\phi(x)}\psi(x)\,dx,$$
where $\lambda > 0$, $\phi : \mathbb{R}^d \to \mathbb{R}$, and $\psi : \mathbb{R}^d \to \mathbb{C}$. In this case, we are interested in the asymptotic behavior of $I(\lambda)$ as $\lambda \to \infty$ (think of λ as time). This is covered in Chapter 8 of Stein's Harmonic Analysis textbook.

2. Second kind:
$$(T_\lambda f)(x) = \int e^{i\lambda\phi(x,y)}K(x,y)f(y)\,dy,$$
where $\lambda > 0$, $\phi : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$, $K : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{C}$, and $f : \mathbb{R}^d \to \mathbb{C}$. We are interested in the asymptotic behavior of the norm of T_λ as $\lambda \to \infty$. This is covered in Chapter 9 of Stein's Harmonic Analysis textbook.

In this class, we'll discuss oscillatory integrals of the first kind, first for $d = 1$, where we will develop a more complete theory.

Proposition 23.1. *Let $\phi : \mathbb{R} \to \mathbb{R}$, $\psi : \mathbb{R} \to \mathbb{C}$ be smooth functions. Assume* $\operatorname{supp}\psi \subseteq (a,b)$ *(nonempty interval), and suppose $\phi'(x) \neq 0$ for all $x \in [a,b]$. Then $I(\lambda) = \int_a^b e^{i\lambda\phi(x)}\psi(x)\,dx$ satisfies $|I(\lambda)| \lesssim_N \lambda^{-N}$ for all $N \geq 0$.*

Proof. We use integration by parts. Write
$$e^{i\lambda\phi(x)} = \frac{1}{i\lambda\phi'(x)}\frac{d}{dx}(e^{i\lambda\phi(x)}).$$

Integrating by parts,
$$I(\lambda) = \int_a^b \frac{d}{dx}(e^{i\lambda\phi(x)})\left(\frac{\psi(x)}{i\lambda\phi'(x)}\right)\,dx = \int_a^b e^{i\lambda\phi(x)}\left[-\frac{d}{dx}\frac{\psi(x)}{i\lambda\phi'(x)}\right]\,dx.$$

Let
$$(Df)(x) = \frac{1}{i\lambda\phi'(x)}\frac{df}{dx}(x).$$

The transpose of D is
$${}^tDf(x) = -\frac{d}{dx}\left(\frac{f(x)}{i\lambda\phi'(x)}\right).$$

Note that
$$D^N(e^{-\lambda\phi}) = e^{i\lambda\phi} \qquad \forall N \geq 1.$$

Now

$$\begin{aligned} I(\lambda) &= \int_a^b D^N(e^{i\lambda\phi(x)})\psi(x)\,dx \\ &= \int_a^b e^{i\lambda\phi(x)}({}^tD)^N\psi(x)\,dx \\ &= \int_a^b e^{i\lambda\phi(x)}\left[-\frac{d}{dx}\frac{1}{i\lambda\phi'(x)}\right]^N. \end{aligned}$$

We get

$$|I(\lambda)| \lesssim \lambda^{-N}\sum_{k=0}^N \sum_{\substack{\beta+\alpha_1+\cdots+\alpha_k=N\\ \alpha_j\ge 1}} \left\| \frac{\psi^{(\beta)}\phi^{(1+\alpha_1)}\dots\phi^{(1+\alpha_k)}}{(\phi')^{N+k}} \right\|_{L^1(a,b)} \lesssim_N \lambda^{-N}. \qquad \square$$

Remark 23.1. If ψ is not compactly supported inside (a,b), then we don't expect better than λ^{-1} decay.

$$\int_a^b e^{i\lambda x}\,dx = \frac{e^{i\lambda b}-e^{i\lambda a}}{i\lambda}.$$

23.2 The Van der Corput lemma

Proposition 23.2 (Van der Corput lemma)**.** *Let $\phi:\mathbb{R}\to\mathbb{R}$ be smooth. Fix $k\ge 1$, and assume $|\phi^{(k)}(x)|\ge 1$ for all $x\in[a,b]$. If $k=1$, assume also that ϕ' is monotonic on $[a,b]$. Then $I(\lambda)=\int_a^b e^{i\lambda\phi(x)}\,dx$ satisfies $|I(\lambda)|\le c_k\lambda^{-1/k}$, where c_k is independent of ϕ,λ,a,b.*

Remark 23.2. If $k=1$, the assumption $|\phi'(x)|\ge 1$ is not sufficient to get the claim. Set $\lambda=1$. Then

$$|I(\lambda)| = \left|\int_a^b e^{i\phi(x)}\,dx\right| \ge \left|\int_a^b \cos(\phi(x))\,dx\right|.$$

If ϕ' is large on $\{x:\cos(\phi(x))<0\}$ and small on $\{x:\cos(\phi(x))>0\}$, then $|\{x:\cos(\phi(x))<0\}| \ll |\{x:\cos(\phi(x))>0\}|$. In particular, $|I(\lambda)| \xrightarrow{b\to\infty} \infty$.

Let's prove the Van der Corput lemma.

Proof. We argue by induction on k. First, let $k=1$. Then

$$\begin{aligned} I(\lambda) &= \int_a^b e^{i\lambda\phi(x)}\,dx \\ &= \int_a^b \frac{1}{i\lambda\phi'(x)}\frac{d}{dx}(e^{i\lambda\phi(x)})\,dx \end{aligned}$$

$$= \frac{e^{i\lambda\phi(b)}}{i\lambda\phi'(b)} - \frac{e^{i\lambda\phi(a)}}{i\lambda\phi'(a)} - \int_a^b e^{i\lambda\phi(x)} \frac{d}{dx}\left(\frac{1}{i\lambda\phi'(x)}\right) dx.$$

So

$$|I(\lambda)| \leq \frac{2}{\lambda} + \frac{1}{\lambda}\int_a^b \left|\frac{d}{dx}\frac{1}{\phi'(x)}\right| dx$$

Since ϕ' is monotonic,

$$\begin{aligned}
&= \frac{2}{\lambda} + \frac{1}{\lambda}\left|\int_a^b \frac{d}{dx}\frac{1}{\phi'(x)}\, dx\right| \\
&= \frac{2}{\lambda} + \frac{1}{\lambda}\underbrace{\left|\frac{1}{\phi'(b)} - \frac{1}{\phi'(a)}\right|}_{\leq 1} \\
&\leq \frac{3}{\lambda}.
\end{aligned}$$

So $c_1 = 3$.

For the inductive step, assume the claim holds for some $k \geq 1$. Assume $|\phi^{(k+1)}(x)| \geq 1$ for all $x \in [a, b]$. Replacing ϕ by $-\phi$ if necessary, we may assume that $\phi^{(k+1)}(x) \geq 1$ for all $x \in [a, b]$. So $\phi^{(k)}$ is increasing on $[a, b]$. Then there exists at most one point $c \in [a, b]$ such that $\phi^{(k)}(c) = 0$. We have two cases:

1. $\exists c \in [a, b]$ such that $\phi^{(k)}(c) = 0$: Since $\phi^{(k)}$ grows at least linearly, there is a δ such that $|\phi^{(k)}(x)| \geq \delta$ for all $x \in [a, b] \setminus (c - \delta, c + \delta)$. Then

$$\left|\frac{d^k}{dx^k}\phi(\delta^{-1/k}x)\right| \geq 1 \qquad \forall \delta^{-1/k}x \in [a, b] \setminus (c - \delta, c + \delta).$$

Then

$$I(\lambda) = \int_a^{c-\delta} e^{i\lambda\phi(x)}\, dx + \int_{c-\delta}^{c+\delta} e^{i\lambda\phi(x)}\, dx + \int_{c+\delta}^b e^{i\lambda\phi(x)}\, dx.$$

Using the change of variables $x = \delta^{-1/k}y$,

$$\left|\int_a^{c-\delta}\right| e^{i\lambda\phi(x)}\, dx = \left|\int_{\delta^{1/k}a}^{\delta^{1/k}(c-\delta)} e^{i\lambda\phi(\delta^{-1/k}y)}\delta^{-1/k}\, dy\right| \leq c_k(\delta\lambda)^{-1/k}$$

by the inductive hypothesis. Similarly,

$$\left|\int_{c+\delta}^b e^{i\lambda\phi(x)}\, dx\right| \leq c_k(\lambda\delta^{-1/k}.$$

For the remaining term, we have

$$\left|\int_{c-\delta}^{c+\delta} e^{i\lambda\phi(x)}\,dx\right| \leq 2\delta.$$

We get

$$I(\lambda)| \leq 2c_k(\lambda\delta)^{-1/k} + 2\delta.$$

Now choose δ such that $(\lambda\delta)^{-1/k} = \delta \implies \delta = \lambda^{-1/(k+1)}$. Then

$$|I(\lambda)| \leq \underbrace{2(c_k+1)}_{c_{k+1}}\lambda^{-1/(k+1)}.$$

2. $\phi^{(k)}(x) \neq 0$ for all $x \in [a,b]$: If $\phi^{(k)}(a) > 0$,

$$|I(\lambda)| \leq \left|\int_a^{a+\delta} e^{i\lambda\phi(x)}\,dx\right| + \left|\int_{a+\delta}^{b} e^{i\lambda\phi(x)}\,dx\right| \leq \delta + c_k(\lambda\delta)^{-1/k}$$

as in the previous case. Setting $\delta = \lambda^{-1/(k+1)}$, we get

$$|I(\lambda)| \leq (c_k+1)\lambda^{-1/(k+1)}.$$

Similarly, if $\phi^{(k)}(a) < 0$, then $\phi^{(k)}(b) < 0$. So we split

$$|I(\lambda)| \leq \left|\int_a^{b-\delta} e^{-\lambda\phi(x)}\,dx\right| + \left|\int_{b-\delta}^{b} e^{i\lambda\phi(x)}\,dx\right| \leq c_k(\delta\delta)^{-1/k} + \delta \leq (c_k+1)\delta^{-1/(k+1)}.$$

□

Corollary 23.1. *Let $\phi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R} \to \mathbb{C}$ be smooth. Fix $k \geq 1$, and assume $|\phi^{(k)}(x)| \geq 1$ for all $x \in [a,b]$. If $k = 1$, assume also that ϕ' is monotonic. Then*

$$I(\lambda) = \int_a^b e^{i\lambda\phi(x)}\psi(x)\,dx$$

satisfies

$$|I(\lambda)| \leq c_k\lambda^{-1/k}\left[|\psi(b)| + \int_a^b |\psi'(x)|\,dx\right].$$

Proof. Write

$$I(\lambda) = \int_a^b \psi(x)\left(\frac{d}{dx}\int_a^x e^{i\lambda\phi(y)}\,dy\right)\,dx$$

Using integration by parts,

$$= \psi(b)\int_a^b e^{i\lambda\phi(y)}\,dy - \int_a^b \psi'(x)\cdot\left(\int_a^x e^{i\lambda\phi(y)}\,dy\right)\,dx.$$ □

24 Estimating Oscillatory Integrals With Stationary Phase

24.1 Estimation in the 1 dimensional case

Proposition 24.1 (stationary phase)**.** *Assume $\phi : \mathbb{R} \to \mathbb{R}$ is smooth and has a non-degenerate critical points at x_0; that is, $\phi'(x_0) = 0$ and $\phi''(x_0) \neq 0$. Assume $\psi : \mathbb{R} \to \mathbb{C}$ is smooth and supported in a sufficiently small neighborhood of x_0. Then*

$$\begin{aligned} I(\lambda) &= \int e^{i\lambda\phi(x)}\psi(x)\,dx \\ &= e^{i\lambda\phi(x_0)}\psi(x_0)\sqrt{2\pi}e^{i(\pi/4)\operatorname{sgn}(\phi''(x_0))}|\phi''(x_0)|^{-1/2}|^{-1/2}\lambda^{-1/2} + O(\lambda^{-3/2}) \end{aligned}$$

as $\lambda \to \infty$.

Remark 24.1. If we are not interested in the coefficient of the leading order term, then we can argue as follows: Let $a \in C_c^\infty$ be such that

$$a(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| > 2 \end{cases}$$

and decompose

$$I(\lambda) = I_1(\lambda) + I_2(\lambda),$$
$$I_1(\lambda) = \int e^{i\lambda\phi(x)}\psi(a)a(\lambda^{1/2}(x - x_0))\,dx.$$

Then

$$\begin{aligned} |I_1(\lambda)| &\leq \|\psi\|_\infty \int |a(\lambda^{1/2}(x - x_0))|\,dx \\ &\leq \|\psi\|_\infty \|a\|_\infty \cdot \lambda^{1/2}, \end{aligned}$$

$$I_2(\lambda) = \int e^{i\lambda\phi(x)}\psi(x)[1 - a(\lambda^{1/2}(x - x_0))]\,dx.$$

Note that $\operatorname{supp}(\psi(x)[1 - a(\lambda^{1/2}(x - x_0))]) \subseteq \{\lambda^{-1/2} \leq |x - x_0| \lesssim_\psi 1\}$. If $\operatorname{supp}\psi$ is such that $\phi'(x) \neq 0$ for $x \in (\operatorname{supp}\psi) \setminus \{x_0\}$, then integration by parts gives

$$|I_2(\lambda)| \lesssim_m \lambda^{-m} \qquad \forall m \geq 0.$$

Proof. Write

$$\phi(x) = \phi(x_0) + \cancelto{0}{\phi'(x_0)(x - x_0)} + \frac{\phi''(x_0)}{2}(x - x_0)^2 + O(|x - x_0|^3).$$

Rewrite this as

$$\phi(x) - \phi(x_0) = \frac{\phi''(x_0)}{2}(x - x_0)^2[1 + \eta(x)],$$

where $\eta(x) = O(|x - x_0|)$. Let U be a small neighborhood of x_0 such that

1. $|\eta(x)| < 1$ for all $x \in U$
2. $\phi'(x) \neq 0$ for all $x \in U \setminus \{x_0\}$.

Assume $\operatorname{supp} \psi \subseteq U$. Change variables to $y(x) = (x - x_0)\sqrt{1 + \eta(x)}$. This is a diffeomorphism from U to a neighborhood of $y = 0$. Then

$$I(\lambda) = e^{i\lambda\phi(x_0)} \int e^{i\lambda[\phi(x)-\phi(x_0)]} \psi(x)\,dx$$
$$= e^{i\lambda\phi(x_0)} \int e^{i\lambda(\phi''(x_0)/2)y^2} \widetilde{\psi}(y)\,dy,$$

where $\widetilde{\psi} \in C_c^\infty$ is supported in a neighborhood of $y = 0$ and $\widetilde{\psi}(0) = \psi(x_0)$.

Let $\widetilde{\widetilde{\psi}} \in C_c^\infty$ be such that $\widetilde{\widetilde{\psi}} = 1$ on $\operatorname{supp} \widetilde{\psi}$. Let $\widetilde{\lambda} = \frac{\lambda\phi''(x_0)}{2}$. Then

$$I(\lambda) = e^{i\lambda\phi(x_0)} \int e^{i\widetilde{\lambda}y^2} e^{-y^2} [e^{y^2} \widetilde{\psi}(y)] \widetilde{\widetilde{\psi}}(y)\,dy$$

Using a Taylor expansion, we write

$$e^{y^2} \widetilde{\psi}(y) = \sum_{j=0}^{N} a_j y^j + y^{N+1} R_N(y), \qquad R_N(y) = \frac{1}{N!} \int_0^1 (1-t)^N \frac{d^{N+1}}{dy^{N+1}} [e^{|\cdot|^2} \widetilde{\psi}](ty)\,dt.$$

This leads us to consider 3 terms:

$$\mathrm{I} = e^{i\lambda\phi(x_0)} \sum_{j=0}^{N} a_j \int e^{i\widetilde{\lambda}y^2} e^{-y^2} y^j\,dy,$$

$$\mathrm{II} = e^{i\lambda\phi(x_0)} \int e^{i\widetilde{\lambda}y^2} e^{-y^2} P_N(y) [\widetilde{\widetilde{\psi}}(y) - 1]\,dy,$$

$$\mathrm{III} = e^{i\lambda\phi(x_0)} \int e^{i\widetilde{\lambda}y^2} e^{-y^2} y^{N+1} R_N 9y) \widetilde{\widetilde{\psi}}(y)\,dy.$$

Since $\frac{d}{dy} e^{i\widetilde{\lambda}y^2}$, we can pull off a factor of y using integration by parts. By picking N to be large enough, we can get as much decay in III as we want.

Let's look at I. Note that the terms with j odd vanish. Consider $j = 0$ and note that $a_0 = \psi(x_0)$. The contribution is

$$e^{i\lambda\phi(x_0)} \psi(x_0) \int e^{i\widetilde{\lambda}y^2} e^{-y^2}\,dy = e^{i\lambda\phi(x_0)} \psi(x_0) (1 - i\widetilde{\lambda})^{1/2} \sqrt{\pi}.$$

To see what happens when $\lambda \to \infty$, write $1 - i\widetilde{\lambda} = re^{i\sigma}$, where $r = \sqrt{1 + \widetilde{\lambda}^2}$ and $\tan\sigma = -\widetilde{\lambda}$. Then $(1 - i\widetilde{\lambda})^{-1/2} = r^{-1/2} e^{-i\sigma/2}$. Then

$$r^{-1/2} = \left(|\widetilde{\lambda}| \sqrt{1 + \frac{1}{\widetilde{\lambda}^2}} \right)^{-1/2}$$

$$= |\widetilde{\lambda}|^{-1/2}\left(1+\frac{1}{\widetilde{\lambda}^2}\right)^{-1/4}$$
$$= \left(\frac{\lambda|\phi''(x_0)|}{2}\right)^{-1/2}\left(1+O(\lambda^{-2})\right)$$
$$= \left(\frac{2}{\lambda|\phi''(x_0)|}\right)^{1/2}+O(\lambda^{-5/2}),$$

$$\tan\sigma = -\widetilde{\lambda} = -\frac{\lambda\phi''(x_0)}{2} \xrightarrow{\lambda\to\infty} -\operatorname{sgn}(\phi''(x_0))\cdot\infty.$$

So $\sigma \to -\operatorname{sgn}(\phi''(x_0))\cdot\frac{\pi}{2}$, and we get

$$e^{i\lambda\phi(x_0)}\psi(x_0)\sqrt{\frac{2\pi}{\lambda|\phi''(x_0)|}}e^{\operatorname{sgn}(\phi''(x_0))\pi/4}+O(\lambda^{-5/2}).$$

For $j \geq 2$ even,

$$\left|\int e^{i\widetilde{\lambda}y^2}e^{-y^2}y^j\,dy\right| = \left|(1-i\widetilde{\lambda})^{-1/2-j/2}\int e^{-y^2}y^j\,dy\right| \lesssim \lambda^{-(j+1)/2} \lesssim \lambda^{-3/2}.$$

Now consider II. Note that $y \mapsto e^{-y^2}P_N(y)[\widetilde{\widetilde{\psi}}(y)-1]$ is supported away from the origin. Integration by parts gives

$$|\,\mathrm{II}\,| \lesssim_m \lambda^{-m} \qquad \forall m \geq 0.$$

Consider III. Decompose $\mathrm{III} = \mathrm{III}_1 + \mathrm{III}_2$, where

$$\mathrm{III}_1 = e^{i\lambda\phi(x_0)}\int e^{i\widetilde{\lambda}y^2}e^{-y^2}y^{N+1}R_N(y)\widetilde{\widetilde{\psi}}(y)a(y/\varepsilon)\,dy.$$

Then

$$|\,\mathrm{III}_1\,| \lesssim \int_{|y|\leq\varepsilon}|y|^{N+1}\,dy \lesssim \varepsilon^{N+2}.$$

The other term is

$$\mathrm{III}_2 = e^{i\lambda\phi(x_0)}\int e^{i\widetilde{\lambda}y^2}y^{N+1}[1-a(y/\varepsilon]b(y)\,dy,$$

where $b(y) = e^{-y^2}R_N(y)\widetilde{\widetilde{\psi}}(y)$. Integration by parts gives

$$\mathrm{III}_2 = e^{i\lambda\phi(x_0)}\int e^{i\widetilde{\lambda}y^2}\cdot\left(-\frac{d}{dy}\frac{1}{2i\widetilde{\lambda}y}\right)^m[y^{N+1}(1-ay/\varepsilon b(y)]\,dy,$$

so

$$|\,\mathrm{III}_2\,| \lesssim_m \frac{1}{\lambda^m}\sum_{k=0}^{m}\sum_{\alpha_1+\alpha_2+\alpha_3=m-k}\left\|\frac{y^{N+1-\alpha_1}\varepsilon^{-\alpha_2}a^{(\alpha_2)}(y/\varepsilon)b^{(\alpha_3)}(y)}{y^{m+k}}\right\|_{L^1}$$

$$
\begin{aligned}
&\lesssim_m \frac{1}{\lambda^m} \int_{|y|\geq\varepsilon} |y|^{N+1-2m}\, dy \\
&\lesssim \frac{\varepsilon^{N+2-2m}}{\lambda^m}
\end{aligned}
$$

if $m > \frac{N+2}{2}$. Now choose ε such that

$$
\varepsilon^{N+2} = \frac{\varepsilon^{N+2-2m}}{\lambda^m} \iff \varepsilon = \lambda^{-1/2}
$$

to get $|\,\mathrm{III}\,| \lesssim \lambda^{-(N+2)/2} \lesssim \lambda^{-3/2}$ if $N \geq 1$. $\square$

25 Oscillatory Integrals in Higher Dimensions

25.1 Nonstationary phase

Here is the case of nonstationary phase.

Proposition 25.1. *Let $\phi : \mathbb{R}^d \to \mathbb{R}$, $\psi : \mathbb{R}^d \to \mathbb{C}$ be smooth. Assume* $\operatorname{supp}\psi$ *is compact and $\|\nabla\phi(x)| \neq 0$ for all $x \in \operatorname{supp}\psi$. Then $I(\lambda) = \int e^{i\lambda\phi(x)}\psi(x)\,dx$ satisfies*

$$|I(\lambda)| \lesssim_m \lambda^{-m} \qquad \forall m \geq 0.$$

Proof. As in the 1 dimensional case, we use integration by parts. We write

$$e^{i\lambda\phi(x)} = \frac{\nabla\phi(x)}{i\lambda|\nabla\phi(x)|^2} \cdot \nabla(e^{i\lambda\phi(x)}).$$

Then

$$I(\lambda) = \int e^{i\lambda\phi(x)}\nabla \cdot \left[\frac{\nabla\phi(x)}{i\lambda|\nabla\phi(x)|^2}\psi(x)\right] dx,$$

so

$$|I(\lambda)| \lesssim \lambda^{-1},$$

where the implicit constant depends on the C^2 norm of ϕ and the C^1 norm of ψ. Now iterate. □

There is an equivalent of Van der Corput's lemma.

Proposition 25.2. *Let $\phi : \mathbb{R}^d \to \mathbb{R}$, $\psi : \mathbb{R}^d \to \mathbb{C}$ be smooth. Assume ψ is compactly supported and $|D^\alpha\phi(x)| \geq 1$ for al $x \in \operatorname{supp}\psi$ for some $\alpha \in \mathbb{N}^d$ with $|\alpha| \geq 1$. Then $I(\lambda) = \int e^{i\lambda\phi(x)}\psi(x)\,dx$ satisfies*

$$|I(\lambda)| \leq C(|\alpha|, \phi)\lambda^{-1/|\alpha|}[\|\psi\|_\infty + \|\nabla\psi\|_1].$$

Remark 25.1. This is worse than the previous proposition when $|\alpha| = 1$. We will also beat it when $|\alpha| = 2$, so we will not actually prove it.

25.2 Stationary phase and Moore's change of variables lemma

Here is the case of stationary phase.

Proposition 25.3 (stationary phase)**.** *Let $\phi : \mathbb{R}^d \to \mathbb{R}$ be smooth, and assume ϕ has a nondegenerate critical point at x_0; that is, $\nabla\phi(x_0) = 0$, but* $\det\left[\frac{\partial^2\phi}{\partial x_i x_j}\right]_{1\leq i,j\leq d}(x_0) \neq 0$.

Assume that $\psi : \mathbb{R}^d \to \mathbb{C}$ is smooth and supported in a sufficiently small neighborhood of x_0. Then

$$\begin{aligned} I(\lambda) &= \int e^{i\lambda\phi(x)}\psi(x)\,dx \\ &= e^{i\lambda\phi(x_0)}\psi(x_0)(2\pi i)^{d/2}\lambda^{-d/2}(\det[D^2\phi(x_0)])^{-1/2} + O(\lambda^{-d/2-1}) \end{aligned}$$

as $\lambda \to \infty$.

Remark 25.2. If we just aim for the correct decay order (and not the precise coefficient), we argue as follows: Let $a : \mathbb{R}^d \to \mathbb{R}$ be a cutoff with

$$a(x) = \begin{cases} 1 & |x| \le 1 \\ 0 & |x| \ge 2 \end{cases}$$

and decompose $I(\lambda) = I_1(\lambda) + I_2(\lambda)$, where

$$I_1(\lambda) = \int e^{i\lambda\phi(x)}\psi(x)a(\lambda^{1/2}(x - x_0))\,dx.$$

Then

$$|I_1(\lambda)| \lesssim \lambda^{-d/2}.$$

Integration by parts gives

$$|I_2(\lambda)| \lesssim_m \lambda^{-m} \qquad \forall m \ge 0.$$

Lemma 25.1 (Morse)**.** *If x_0 is a nondegenerate critical point of a smooth function $\phi : \mathbb{R}^d \to \mathbb{R}$, then there exists a smooth change of variables $x \mapsto y(x)$ such that $y(x_0) = 0$, $\frac{\partial y}{\partial x}(x_0) = \mathrm{Id}$, and*

$$\phi(x) - \phi(x_0) = \sum_{j=1}^{d} \frac{1}{2}\lambda_j y_j^2,$$

where $\lambda_1, \dots, \lambda_d$ are the eigenvalues of $D^2\phi(x_0)$.

Proof. Performing an orthogonal change of variables, we may assume that $D^2\phi(x_0) = \mathrm{diag}(\lambda_1, \dots, \lambda_d)$. By Taylor expansion,

$$\phi(x) = \phi(x_0) + \cancelto{0}{\nabla\phi(x_0)} \cdot (x - x_0) + \int_0^1 (1-t)\frac{d^2}{dt^2}[\phi(x_0 + t(x - x_0))]\,dt.$$

So

$$\phi(x) - \phi(x_0) = \int_0^1 (1-t)\frac{d}{dt}[(x - x_0)\cdot\nabla\phi(x_0 + t(x - x_0))]\,dt$$

$$
\begin{aligned}
&= \sum_{i,j\geq 1} \int_0^1 (1-t)(x-x_0)_i (x-x_0)_j \frac{\partial^2 \phi}{\partial x_i \partial x_j}(x_0 + t(x-x_0))\, dt \\
&= \sum_{i,j\geq 1} (x-x_0)_i (x-x_0)_j m_{i,j}(x),
\end{aligned}
$$

where $m_{i,j}(x) = \int_0^1 (1-t) \frac{\partial^2\phi}{\partial x_i \partial x_j}(x_0 + t(x-x_0))\, dt$. Note that the $m_{i,j}$ are smooth, $m_{i,j}(x) = m_{j,i}(x)$, and $m_{i,j}(x_0) = \frac{1}{2}\frac{\partial^2\phi}{\partial x_i \partial x_j}(x_0)$. So

$$
[m_{i,j}(x_0)]_{1\leq i,j\leq d} = \frac{1}{2}\operatorname{diag}(\lambda_1, \ldots, \lambda_d).
$$

We argue inductively. Assume

$$
\phi(x) - \phi(x_0) = \frac{1}{2}\lambda_1 y_1^2 + \cdots + \frac{1}{2}\lambda_{r-1} y_{r-1}^2 + \sum_{i,j\geq r} \widetilde{m}_{i,j}(y) y_i y_j
$$

for some $1 \leq r \leq d$, where $y(x_0)$ =0, $\frac{\partial y}{\partial x}(x_0) = \mathrm{Id}$, and $\widetilde{m}_{i,j} = \widetilde{m}_{j,i}$. We know that $D^2[\mathrm{RHS}(x)]_{x=x_0} = \operatorname{diag}(\lambda_1, \ldots, \lambda_d)$. Then

$$
\begin{aligned}
\frac{\partial^2}{\partial x_i \partial x_j}\left(\frac{1}{2}\lambda_k y_k^2\right)\bigg|_{x=x_0} &= \left[\lambda_k \frac{\partial y_k}{\partial x_i}\frac{\partial y_k}{\partial x_j} + \lambda_k y_k \frac{\partial^2 y_k}{\partial x_i \partial x_j}\right]_{x=x_0} \\
&= \lambda_k \delta_{i,k}\delta_{j,k}.
\end{aligned}
$$

So

$$
\left[D^2\left(\sum_{i,j\geq r} \widetilde{m}_{i,j}(y) y_i y_j\right)\right](x_0) = \operatorname{diag}(0, \ldots, 0, \lambda_r, \ldots, \lambda_d).
$$

We now have

$$
\frac{\partial^2}{\partial x_k \partial x_\ell}\left(\sum_{i,j\geq r} \widetilde{m}_{i,j}(y) y_i, y_j\right)\Bigg|_{x=x_0} = \sum_{i,j\geq r} \widetilde{m}_{i,j}(0)\left(\delta_{k,i}\delta_{\ell,j} + \delta_{\ell,i}\delta_{k,j}\right).
$$

This tells us that

$$
[\widetilde{m}_{i,j}(0)]_{r\leq i,j\leq d} = \frac{1}{2}\operatorname{diag}(\lambda_r, \ldots, \lambda_d)
$$

Change variables as follows:

$$
\begin{cases}
y_j' = y_j & j \neq r \\
y_r' = \sqrt{\frac{\widetilde{m}_{r,r}(y)}{\lambda_r/2}}\left(y_r + \sum_{j\geq r+1} \frac{\widetilde{m}_{j,r}(y)}{\widetilde{m}_{r,r}(y)} y_j\right).
\end{cases}
$$

We need to show that this is a diffeomorphism with $y'(x_0) = 0$, $\frac{\partial y'}{\partial x}|_{x=x_0} = \mathrm{Id}$, and

$$\phi(x) = \phi(x_0) = \frac{1}{2}\lambda_1 y_1^2 + \cdots + \frac{1}{2}\lambda_r (y_r')^2 + \sum_{i,j\geq r+1} \widetilde{\widetilde{m}_{i,j}}(y) y_i y_j.$$

We have $y'(x_0) = 0$ because each y_i is 0 at x_0. For $j \neq r$,

$$\left.\frac{\partial y_j'}{\partial x_i}\right|_{x-x_0} = \delta_{i,j},$$

so

$$\left.\frac{\partial y_r'}{\partial x_i}\right|_{x=x_0} = \sqrt{\frac{\widetilde{m}_{r,r}(0)}{\lambda_r/2}} \left(\delta_{i,r} + \sum_{j\geq r+1} \cancelto{0}{\frac{\widetilde{m}_{j,r}(0)}{\widetilde{m}_{r,r}(0)}} \delta_{j,i} \right) = \delta_{i,r}$$

Now we have

$$\begin{aligned}
\sum_{i,j\geq r} \widetilde{m}_{i,j}(y) y_i y_j - \frac{1}{2}\lambda_r (y_r')^2 &= \sum_{i,j\geq r} \widetilde{m}_{i,j}(y) y_i y_j \\
&\quad - \widetilde{m}_{r,r} \left(y_r^2 + 2 \sum_{j\geq r+1} \frac{\widetilde{m}_{j,r}(y)}{\widetilde{m}_{r,r}(y)} y_i, y_r \right. \\
&\quad \left. + \sum_{i,j\geq r+1} \frac{\widetilde{m}_{i,r}(y)\widetilde{m}_{j,r}(y)}{\widetilde{m}_{r,r}(y)\widetilde{m}_{r,r}(y)} y_i, y_j \right) \\
&= \sum_{i,j\geq r+1} \underbrace{\left[\widetilde{m}_{i,j}(y) - \frac{\widetilde{m}_{i,r}(y)\widetilde{m}_{j,r}(y)}{\widetilde{m}_{r,r}(y)} \right]}_{=\widetilde{\widetilde{m}}_{i,j}(y)} y_i, y_j.
\end{aligned}$$

This completes the proof. $\square$